Dynamic Risk of CDOs

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Abstract

The current financial crisis is a remarkable example of underestimation of systematic risk. To that end, we investigate risk characteristics of synthetic CDOs in detail. This requires a fully dynamic approach including the business cycle. We find significant dependence of tranche default probability (PD) and tranche loss given default (LGD) on the credit cycle state. Furthermore, we show the need of modeling LGD as stochastic and PD dependent process.

Contents

1 Introduction

The current financial crisis is a result of several factors, one of the most important being ill-rated structured finance products. One of the reasons for the agencies' failure seems to be underestimation of systematic risk. Among many factors which may be influential in a rating process two things appear most important in our view: disregard of the increased systematic risk sensitivity of CDOs in comparison with bonds as well as through-the-cycle modeling.

In this article, we shall address both issues. First, we develop a simple and consistent extension of the Gaussian 1-factor model which allows for the state of default-relevant systematic factors. Based on this we study a series of risk characteristics of CDOs and show the model risks of disregarding the evolution of the business cycle as well as LGD volatility.

In the discrete-time literature[1], some models can already be found allowing for systematic risks with persistence. One group of works is based on the Gaussian 1-factor model assuming an autoregressive process for the systematic factor (Koopman, Lucas, and Klaassen, 2005; McNeil and Wendin, 2007; Lamb, Perraudin, and van Landschoot, 2008; Lamb and Perraudin, 2008). Another part of the literature models the systematic factor as a Markov chain (Giampieri, Davis, and Crowder, 2005; Bruche and Gonzalez-Aguado, 2008; Pederzoli and Torricelli, 2005). Of all these articles, only Lamb, Perraudin, and van Landschoot (2008) consider structured finance products or CDOs in particular.

Besides, there are several articles on CDO risk. Fender and Kiff (2005) study CDO rating methodology and Fender, Tarashev, and Zhu (2008) examine the sensitivity of tranche rating to obligor rating changes or correlation changes. The paper of Gibson (2004) is concerned with several aspects of CDO risk, mostly related to spread sensitivity. These articles do not examine the underlying models formally and in-depth. Another part of the literature is concerned with capital allocation rules for portfolios containing CDO tranches (Gordy and Jones, 2003; Pykhtin and Dev, 2002, 2003). These contributions are largely based on LHP approximations which is not our focus. Furthermore, this article is limited to risk assessment or rating of CDOs based on real world default risk measures.

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[1]Autocorrelation in continuous time is intrinsically given with diffusions. We will not pursue these models here.

To summarize, the major contribution of this article is an in-depth analysis of the risk particularities of CDO tranches. We show that there is always a strong dependence of both tranche PD (probability of default) as well as LGD (loss given default) on the systematic factor. This is a very important fact but hardly ever studied explicitly in the literature. We show the relevance of dynamic modeling when a security is highly exposed to systematic risks. To that end, we analyze the evolution of tranche risk measures in the course of time. The latter implies dependence on the state of the credit cycle and we show that disregarding this may lead to seriously biased risk estimates. We introduce a dynamic and factor-conditional representation of tranche PD and LGD which is useful at risk assessment as well as for portfolio integration of securitisations. We are not aware of any prior works in this respect. In addition, because of their potential to induce systematic effects, we investigate and compare different dynamic LGD volatility models and their contributions to tranche risk. We find again that disregard of this may lead to large deviations. Finally, we show the extreme impact of external shocks of average collateral pool PD and LGD similar to those currently observed in the U.S.

The article is organized as follows. In the next section we introduce our dynamic model including several LGD volatility extensions. We show in detail necessary calculations for default threshold calibration. Finally, we briefly introduce some important tranche risk measures. In Section 3 we perform an extensive portfolio study with the results as described above.

2 General Factor Model Setup

In this section we describe the model setup upon which our analyses are based. We consider a Merton-style Gaussian one-factor model as suggested in the Basel II specification. F represents a common systematic factor which affects all obligors' default probabilities.

2.1 Single Period Gaussian 1-Factor Model

In a portfolio of $i = 1, \ldots, n$ obligors default of obligor i is modeled as a threshold event:

$$D_i = \mathbf{1}_{\{R_i < c_i\}} \tag{1}$$

where R_i is a random variable comprising two terms:

$$R_i = \sqrt{\rho} F + \sqrt{1 - \rho} U_i \tag{2}$$

a common (systematic) factor F and an idiosyncratic factor U_i. Both are iid standard normal and so is R_i. D_i is a default indicator which jumps to

unity if R_i falls below c_i.

Now let LGD_i denote loss given default of obligor i as a fraction of his notional N_i. Then loss is given by

$$L = \sum_i \mathbf{1}_{\{D_i=1\}} \cdot \text{EAD}_i \cdot \text{LGD}_i \tag{3}$$

To simplify the dependence structure as well as the calculation of L we make the common assumption of conditional independence with respect to F. This renders the joint probability distribution as the product of the margins $\lambda_i(F)$ which are given by

$$\lambda_i(F) = \Phi\left(\frac{\Phi^{-1}(\lambda_i) - \sqrt{\rho}F}{\sqrt{1-\rho}}\right) \tag{4}$$

2.2 Multiple Periods Gaussian 1-Factor Model

So far the described approach is standard and frequently applied in practice. Our next step is to extend the model to multiple periods. This is most easily done assuming stochastic processes for F and U_i, i.e., F_t and U_{it}. In the absence of any other hypothesis we simply assume that U_{it} is iid standard normal.

However, the systematic term admits more structure. Empirical research shows that default rates have a cyclical behavior and persistence phases. Hence, we specify F_t in one of the most simple ways as first-order autoregressive process AR(1):

$$F_t = \alpha_1 F_{t-1} + \sigma W_t \tag{5}$$

where α_1 and σ are parameters and F_0 is the initial value of the process. Furthermore, we set $\sigma = \sqrt{1-\alpha_1^2}$ so that $F_t \to \mathcal{N}(0,1)$ as t grows. The two first moments of the unconditional process F_t are $\mathbb{E}[F_t] = 0$ and $\mathbb{V}[F_t] = 1$ and given F_0 we have $\mathbb{E}[F_t] = \alpha_1^t F_0$ and $\mathbb{V}[F_t] = \sigma^2 \sum_{j=0}^{t-1} \alpha_1^{2j} = 1 - \alpha_1^{2t}$

Substituting this we obtain for R_{it}

$$R_{it} = \sqrt{\rho}\alpha_1^t F_0 + \sqrt{\rho}\sigma \sum_{j=0}^{t-1} \alpha_1^j W_{t-j} + \sqrt{1-\rho}U_{it} \tag{6}$$

where $i \in N_t$, i.e., the index set of survivors in $t-1$, $N_t = \{i : D_{it'} = 0, t' < t\}$.

In a multi-period setting we evaluate our threshold model once per period among the survivors. The default indicators are defined accordingly as

follows

$$D_{it} = \mathbf{1}_{\{R_{it} < c_{it}\}} \tag{7}$$

and portfolio loss as of time t is defined as

$$L_t = \sum_{t' \leq t} \sum_i D_{it'} \cdot \text{EAD}_i \cdot \text{LGD}_{it'} \tag{8}$$

Unconditional Multi-Period R_{it} Distribution Evaluation of a T period model by means of repeated computation of a single-period model requires to adapt the default thresholds c_{it} to the hazard rate term structure (λ_{it}). Thus, for any t we set

$$\mathbb{P}\big[R_{it} < c_{it} \mid R_{it'} > c_{it'}, t' < t\big] = \lambda_{it} \tag{9}$$

and solve for c_{it}. The marginal and joint distributions of R_{it} and $R_{it'}, t' < t$, are Gaussian. If hazard rates are not conditional upon the state of the credit cycle (i.e., if they are "through-the-cycle"), calibration requires taking expectation with respect to the common factor F. The resulting multivariate normal of $(R_{i1}, \ldots, R_{it})$ has expectation

$$(\mathbb{E}[R_{i1}], \ldots, \mathbb{E}[R_{it}]) = (0, \ldots, 0) \tag{10}$$

and covariances

$$\begin{aligned} Cov\left(R_{it}, R_{it'}\right) &= \mathbb{E}[R_{it}, R_{it'}] - \mathbb{E}[R_{it}]\mathbb{E}[R_{it'}] \\ &= \rho \alpha_1^{|t-t'|} + (1-\rho)^{\mathbf{1}_{\{t=t'\}}} \end{aligned} \tag{11}$$

The above hazard rate based on a multidimensional Gaussian is easily simulated. As a result, we may derive c_{it} consecutively (given $c_{it'}, t' < t$) in a bootstrap fashion[2].

Conditional Multi-Period R_{it} Distribution In the following we shortly anaylze the corresponding conditional distributions, i.e., given F_0 we have

$$(R_{it}, R_{jt'}) \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \tag{12}$$

with mean

$$\boldsymbol{\mu} = \left(\sqrt{\rho}\alpha_1^t f_0, \sqrt{\rho}\alpha_1^{t'} f_0\right) \tag{13}$$

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[2]Inversion is done easily via a one-dimensional root search algorithm.

and covariances

$$Cov\left(R_{it}, R_{jt'}\right) = \mathbb{E}\left[R_{it}, R_{jt'}\right] - \mathbb{E}[R_{it}]\mathbb{E}\left[R_{jt'}\right]$$
$$= \rho\left(\alpha_1^{|t-t'|} - \alpha_1^{|t+t'|}\right) + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}} \tag{14}$$

which implies correlation

$$\rho_{R_{it},R_{jt'}} = \frac{Cov\left(R_{it}, R_{jt'}\right)}{\sqrt{\mathbb{V}[R_{it}]\mathbb{V}\left[R_{jt'}\right]}}$$
$$= \rho\frac{\alpha_1^{|t-t'|}\left(1 - \alpha_1^{2t'}\right)}{\sqrt{\left(1 - \rho\alpha_1^{2t}\right)\left(1 - \rho\alpha_1^{2t'}\right)}} \tag{15}$$

The last correlation term effectively says that keeping t' constant and increasing t implies decreasing correlation. In a one period model any two obligors have correlation ρ while in a multi-period model contemporaneous correlation is

$$\rho_{R_{it},R_{jt}} = \rho\frac{1 - \alpha_1^{2t}}{1 - \rho\alpha_1^{2t}} \tag{16}$$

Letting $t \to \infty$, this correlation approaches ρ. Intertemporal correlation between i and j tends to zero as $|t - t'|$ increases. This is certainly one reason why single-period and multiple-periods models based on the same asset correlation are inconsistent. For any two obligors the correlation is ρ in a single-period model but not so (i.e., lower) in a multiple-periods framework.

This problem is still present even if the variance of F_t is normalized to unity in each period, i.e., by setting $\sigma = 1$ in $t = 1$ and $\sigma = \sqrt{1 - \alpha_1^2}$ in $t > 1$. More precisely, we have

$$(R_{it}, R_{jt'}) \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \tag{17}$$

with mean

$$\boldsymbol{\mu} = \left(\sqrt{\rho}\alpha_1^t f_0, \sqrt{\rho}\alpha_1^{t'} f_0\right) \tag{18}$$

but covariances

$$Cov\left(R_{it}, R_{jt'}\right) = \mathbb{E}\left[R_{it}, R_{jt'}\right] - \mathbb{E}[R_{it}]\mathbb{E}\left[R_{jt'}\right]$$
$$= \rho\alpha_1^{|t-t'|} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}} \tag{19}$$

From the last line it is again evident that intertemporal correlation is always lower than contemporaneous correlation.

Finally, note that normalization implies another model since even if the thresholds are calibrated to match the term structures of i and j (i.e., the margins), we have still Gaussian copulas with different correlations. In other words, both models have the same margins but different dependence structures. To see the great difference of both models consider the case where $\alpha_1 = 1$. The unnormalized model implies zero correlation of $(R_{it}, R_{jt'})$ for any t and t'. However, the normalized model leads to correlation ρ, again for any t and t'. In the first case, all obligors have the same F_0 each period so that volatility is only due to their iid terms U_{it} and $U_{jt'}$. As a result, they are uncorrelated. In the second case, there is only one innovation, namely in $t = 1$. For $t > 1$ we have $\sigma = 0$ and no further innovations are added. As a result, the same systematic innovation influences all obligors in each period (and without attenuation since $\alpha_1 = 1$) and hence, ρ is both contemporaneous and intertemporal correlation.

Deriving (cumulative) portfolio loss L_t at time t again relies on conditional independence on F_t. Let $\tilde{R}_{it} = R_{it} \mid R_{it'} > c_{it'}, t' < t$. Then $\mathbb{P}\left[\tilde{R}_{it} < c_{it}\right] = \lambda_{it}$ but the distribution of $\tilde{R}_{it}$ is not immediate. However, $\tilde{R}_{it} \mid F_t \sim \mathcal{N}(\sqrt{\rho}F_t, 1 - \rho)$ and we obtain

$$\lambda_{it}\left(F_t\right) = \Phi\left(\frac{c_{it} - \sqrt{\rho}F_t}{\sqrt{1 - \rho}}\right) \tag{20}$$

2.3 Stochastic Loss Given Default (LGD)

Applying a static (single-period) model is one questionable approximation frequently made, assuming fixed, deterministic LGDs is another. Hence, our next step is to extend the model for stochastic LGDs.

Common Factor A first approach is based on Duellmann and Trapp (2004) who assume the following latent process for $Y_{it} = \ln\left(\frac{1 - \text{LGD}_{it}}{\text{LGD}_{it}}\right)$

$$Y_{it} = \mu + \tilde{\sigma}\sqrt{\omega_1}F_t + \tilde{\sigma}\sqrt{1 - \omega_1}E_{it} \tag{21}$$

where $i \in D_t$, the index set of defaulted names by the end of t, $D_t = \{i \in N_t : D_{it} = 1\}$ and E_{it} is a name-specific standard normally distributed innovation.

μ and $\tilde{\sigma}$ are linear transformation coefficients and ω_1 controls the influence of F_t. The first two moments are $\mathbb{E}[Y_{it}] = \mu$ and $\mathbb{V}[Y_{it}] = \tilde{\sigma}$ without knowledge of F_0 and $\mathbb{E}[Y_{it} \mid F_0] = \mu + \tilde{\sigma}\sqrt{\omega_1}\alpha^t F_0$ and $\mathbb{V}[Y_{it} \mid F_0] = \tilde{\sigma}^2\left(1 - \omega_1\alpha^{2t}\right)$ given F_0. The latter conditional moments tend to the unconditional ones as t rises. Based on this, LGD arises after logistic transforma-

tion

$$\text{LGD}_{it} = \frac{1}{1 + exp(Y_{it})} \qquad (22)$$

i.e., as $Y_{it} \to -\infty$, $\text{LGD}_{it} \to 1$.

Obviously, this specification assumes that LGD is based on the same latent process as PD.

Separate Factors   There is no definite reason to assume a common factor for both PD and LGD. Hence, we also consider a model where the LGD process is based on a separate factor process.

$$\begin{aligned} F_t &= \alpha_1 F_{t-1} + \sigma_1 W_t \\ G_t &= \alpha_2 G_{t-1} + \sigma_2 V_t \end{aligned} \qquad (23)$$

where $(W_t, V_t) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{\mu} = (0,0)$ and $\boldsymbol{\Sigma}$ is a 2×2 covariance matrix with unit diagonal and off-diagonal ω_2. The latent factor process G_t drives the same process as in the common factor model

$$Y_{it} = \mu + \tilde{\sigma}\sqrt{\omega_1}G_t + \tilde{\sigma}\sqrt{1 - \omega_1}E_{it} \qquad (24)$$

Furthermore, we set again $\sigma_{1,2} = \sqrt{1 - \alpha_{1,2}^2}$. In this setup, both latent processes are related via the correlation of the innovations. More precisely, (F_t, G_t) is multivariate normal with

$$\begin{aligned} \mathbb{E}[F_t, G_t] &= \left(\alpha_1^t F_0, \alpha_2^t G_0\right) \\ \mathbb{V}[F_t, G_t] &= \left(\sigma_1^2 \sum_{j=0}^{t-1} \alpha_1^{2j}, \sigma_2^2 \sum_{j=0}^{t-1} \alpha_2^{2j}\right) = \left(1 - \alpha_1^t, 1 - \alpha_2^t\right) \\ Cov[F_t, G_t] &= \sigma_1 \sigma_2 \omega_2 \sum_{j=0}^{t-1} (\alpha_1 \alpha_2)^j \end{aligned} \qquad (25)$$

For $\omega_2 = 1$ and $\alpha_1 = \alpha_2$ both processes are prefectly correlated.

2.4   Synthetic CDOs

In the last subsections we developed a model for asset pool loss. This paper is concerned with derivatives on an asset pool, specifically synthetic CDOs. In the sequel, we introduce the relevant notation.

Let $0 \leq A_{\text{tr}} < B_{\text{tr}} \leq 1$, denote a percentage interval of the asset pool notional $N = \sum_i N_i$, called "tranche". The CDO is "hit", i.e., incurs losses,

if total cumulative asset pool loss exceeds the lower attachment point A_{tr} of the tranche, i.e., $L_t > A_{\text{tr}}$. A complete default of the CDO occurs if $L_t \geq B_{\text{tr}}$. Formally, CDO tranche loss is given by

$$L_t^{\text{tr}} = \min\left(L_t, B_{\text{tr}}\right) - \min\left(L_t, A_{\text{tr}}\right) \qquad (26)$$

The loss given default of a CDO tranche is given by

$$LGD_t^{\text{tr}} = L_t^{\text{tr}} \mid L_t > A_{\text{tr}} \qquad (27)$$

Several other tranche related risk measures shall be relevant below.

Tranche Hitting Probability

$$p_t^{\text{tr}} = \mathbb{P}[L_t > A_{\text{tr}}] \qquad (28)$$

Tranche Hazard Rate

$$\lambda_t = \frac{\mathbb{P}[L_t > A_{\text{tr}}, L_{t-1} \leq A_{\text{tr}}]}{\mathbb{P}[L_{t-1} \leq A_{\text{tr}}]} \qquad (29)$$

This is actually a probability due to time discretization.

Expected Tranche Loss Given Survival   Above we argued that LGD does also depend on the PD factor process. In order to examine this dependence in a dynamic context we consider

$$\mathbb{E}\left[L_t^{\text{tr}} \mid L_{t-1}^{\text{tr}} = 0\right] \qquad (30)$$

i.e., expected loss in period t given survival until $t-1$. Conditioning on both non-default as of $t-1$ and F_t yields another important measure

$$\mathbb{E}\left[L_t^{\text{tr}} \mid L_{t-1}^{\text{tr}} = 0, F_t = f_t\right] \qquad (31)$$

This expression tells us the mean loss a tranche which has not been hit yet will suffer given the economy is in state f_t then.

Expected Tranche LGD given Survival

$$\mathbb{E}\left[L_t^{\text{tr}} \mid L_{t-1}^{\text{tr}} = 0, L_t^{\text{tr}} > 0\right] \qquad (32)$$

Marginal VaR (MVaR)   Finally, we shall examine MVaR of CDO tranches. Esentially, this is the VaR add-on when a specific tranche is included in a

portfolio. This figure is usually difficult to calculate and it is portfolio dependent. However, assuming an infinitely fine-grained bank/investor portfolio (Pykhtin and Dev, 2002) admits a simple solution as VaR contributions are portfolio-invariant then. Specifically,

$$\mathrm{MVaR}_i(q) = \mathbb{E}\big[L_i \mid \boldsymbol{F}_{F_t}^{-1}(q)\big] \tag{33}$$

where $\boldsymbol{F}_{F_t}^{-1}$ denotes the inverse of the cdf of F_t.

3   Portfolio Study

In this section we study the behavior of tranche PD and tranche LGD of a CDO described in Bluhm and Overbeck (2007). The asset pool comprises $n = 100$ names from several rating classes and PDs ranging from 0.00002 to 0.3235 in the first year (see Table 1). Maturity is $T = 5$ years. Portfolio

Class	#	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
AAA	5	0.0000	0.0000	0.0004	0.0003	0.0005
AA	12	0.0001	0.0002	0.0005	0.0008	0.0010
A	22	0.0004	0.0009	0.0013	0.0017	0.0023
BBB	32	0.0029	0.0057	0.0063	0.0090	0.0090
BB	17	0.0128	0.0271	0.0350	0.0344	0.0318
B	8	0.0624	0.0863	0.0845	0.0752	0.0607
CCC	4	0.3235	0.1478	0.1095	0.0972	0.1260

Table 1: Asset pool described in Bluhm and Overbeck (2007): rating class, number of names, hazard rates for $t = 1, \ldots, 5$.

exposures are homogeneous. We consider two mezzanine tranches in more detail: $\mathrm{Tr}_1 : (A_1, B_1) = (0.05, 0.08)$ and $\mathrm{Tr}_2 : (A_2, B_2) = (0.08, 0.11)$.

We examine six cases in more detail (see Table 2). In the first case LGD is non-stochastic. Case 2 and case 3 are benchmarks for case 1 with fixed LGD. Here, F_0 is assumed standard normal. In addition, case 3 has no autocorrelation. Case 4 to 6 relate to different LGD models. In case four, LGD is stochastic but depends on the same factor F_t as the PD process. Case five is again with stochastic LGD but now we have two separate but correlated factor processes. Finally, in case 6, we investigate whether LGD volatility and time dependence is by itself sufficient to explain higher tranche default risks. This is accomplished by setting the covariance between W_t and V_t to zero, i.e., $\omega_2 = 0$. In addition, case 7 is to show the effect of LGD stress. To that end, μ is set so that $\mathbb{E}[LGD] = 0.6$ instead of 0.5.

3.1   Sensitivity of Tranche PD

One of the most important differences between subordinated defaultable products and regular ones is the dependence on systematic risk. Systematic influence is primarily driven via the default probabilities in the asset pool

	Parameter							
Case	F_0	ρ	α_1	α_2	μ	$\tilde{\sigma}$	ω_1	ω_2
1	$F_0 \in \{-1, 0, +1\}$	0.12	0.80	—	—	—	—	—
2	$F_0 \sim \mathcal{N}(0,1)$	0.12	0.80	—	—	—	—	—
3	$F_0 \sim \mathcal{N}(0,1)$	0.12	0	—	—	—	—	—
4	$F_0 \in \{-1, 0, +1\}$	0.12	0.80	—	0	0.35	0.13	—
5	$F_0 \in \{-1, 0, +1\}$	0.12	0.80	0.80	0	0.35	0.13	0.70
6	$F_0 \in \{-1, 0, +1\}$	0.12	0.80	0.80	0	0.35	0.13	0
7	$F_0 \in \{-1, 0, +1\}$	0.12	0.80	—	−0.405	0.35	0.13	—

Table 2: Parameter Configurations.

which are assumed to depend on F_t. But F_t in turn depends on F_0 and the strength of this relation is controlled by α_1. Besides, systematic influence also unfolds through asset pool LGD. Higher average LGDs in the asset pool accelerate the reduction of subordination. In the following subsections we study these effects in detail.

3.1.1 Initial Factor Dependence

In order to show the relevance of the initial factor F_0 we consider tranche hitting probabilities in the course of time. We confine our analysis to cases with fixed LGD (cases 1, 2, and 3) of the list of model configurations. Below, we will turn to cases with stochastic LGD.

Table 3 shows the simulation results. $F_0 = \pm 1$ represent moderate good and bad factor levels ($\Phi(-1) = 0.158$), respectively, and $F_0 = 0$ is a mean level. In Table **??** we see that hitting probabilities are strongly influenced by F_0. A low (bad) initial value of the economic factor implies high tranche hitting risk and vice versa for a high initial value. The strength of this dependence is remarkable: for tranche 1 and 2 cumulative hitting probabilites in $t = 5$ are, respectively, 5 and more than 10 times as high for $F_0 = +1$ in comparison with $F_0 = -1$. This result, which carries over to other asset pool configurations and tranches, underlines the fact that disregard of the current state in the economic cycle may result in seriously misleading rating estimates for the near future. Why? Increased systematic risk sensitivity means that small movements of the systematic factor have large impact on risk measures. As a result, looking through the cycle conceals the increased variance of risk measures.

These arguments are strengthened considering case 2 and case 3, i.e., when F_0 is random or when the whole process F_t is iid. The results (see Table 3) show significant differences between case 1 on the one hand and case 2 and 3 on the other hand[3]. With a positive initial value F_0 risk measures are at very low levels but with a negative initial value hitting probabilities soar. This is a good example for the importance of stress tests with products having high systematic risk exposure.

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[3]The only exception is case 1 with $F_0 = 0$ which yields similar probabilities as case 3.

	Case 1		
Tranche	$t = 1$	$t = 3$	$t = 5$
$F_0 = -1$			
1	0.0006	0.1807	0.4859
2	0.0000	0.0165	0.1309
$F_0 = 0$			
1	0.0000	0.0472	0.2473
2	0.0000	0.0020	0.0399
$F_0 = +1$			
1	0.0000	0.0078	0.0988
2	0.0000	0.0002	0.0100

	Case 2,3		
Tranche	$t = 1$	$t = 3$	$t = 5$
Case 2			
1	0.0011	0.0933	0.2851
2	0.0000	0.0113	0.0716
Case 3			
1	0.0010	0.0559	0.2533
2	0.0000	0.0022	0.0213

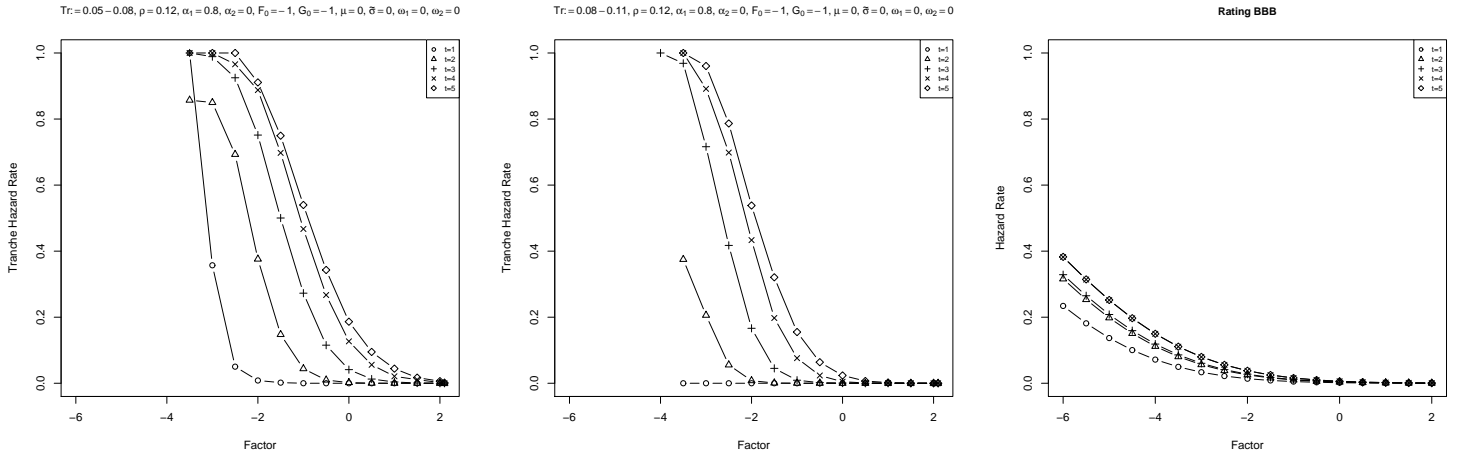**Table 3:** Tranche hitting probability.

Figure 1: Factor sensitivity of tranche hazard rate for tranches Tr_1 and Tr_2 and a bond from rating class BBB.

3.1.2 Current Factor Dependence

Our previous analysis has shown the dependence of tranche hitting probability on the initial PD factor level. But how does tranche default risk in period t depend on F_t? This question is important, for instance, when a CDO has to be integrated in a bond portfolio. Similarly, knowing sensitivity with respect to F_t is a necessary precondition to model CDO loss distributions. Consider the following two "sensitivity diagrams" (Figure 1). The curves represent factor specific tranche hazard rates, i.e., the hitting probability in period t given no default until $t - 1$ and given $F_t = f_t$.

We see that lower levels of F_t imply higher tranche hazard rates. For $t = 1$ tranche hazard rates are identical to tranche hitting probabilities. In the last section we could see how default rates go down from $F_1 = -1$ to $F_1 = +1$. For $t > 1$ tranche hazard rates are hitting probabilities conditional upon no hit until the last period. Similar to $t = 1$ we find rates increasing from zero to unity as the current factor F_t decreases.

Furthermore, the curve shifts to the right for growing t, which is a com-

mon but not a necessary phenomenon[4]. Finally, comparing the right figure with the other two, the difference between tranches and bonds becomes obvious. Tranche hazard rates increase much faster with decreasing factor level than conventional bonds.

3.1.3 Asset Pool LGD Dependence

As a figure based on cumulative loss, tranche PD also depends on asset pool LGDs. As pointed to above, empirical studies provide evidence for bond LGD volatility and default rate dependence. To that end, we shall include the LGD extensions as presented in the last section in our analysis (case 4, 5, and 6 in Table 2). As a reminder, case 1 is fixed LGD while 4-6 are stochastic LGD. Case 4 is with common factor dependence and 5-6 are with a separate LGD factor process (G_t). Case 6 involves no correlation between W_t and V_t.

Table 4 shows the results.

	Case 1				Case 4		
Tranche	$t = 1$	$t = 3$	$t = 5$	Tranche	$t = 1$	$t = 3$	$t = 5$
$F_0 = -1$				$F_0 = -1$			
1	0.0006	0.1807	0.4859	1	0.0035	0.2595	0.5587
2	0.0000	0.0165	0.1309	2	0.0000	0.0395	0.1928
$F_0 = 0$				$F_0 = 0$			
1	0.0000	0.0472	0.2473	1	0.0001	0.0765	0.3006
2	0.0000	0.0020	0.0399	2	0.0000	0.0056	0.0649
$F_0 = +1$				$F_0 = +1$			
1	0.0000	0.0078	0.0988	1	0.0000	0.0145	0.1258
2	0.0000	0.0002	0.0098	2	0.0000	0.0004	0.0171
	Case 5				Case 6		
Tranche	$t = 1$	$t = 3$	$t = 5$	Tranche	$t = 1$	$t = 3$	$t = 5$
$F_0 = -1$				$F_0 = -1$			
1	0.0026	0.2571	0.5569	1	0.0018	0.2441	0.5550
2	0.0000	0.0360	0.1876	2	0.0000	0.0283	0.1704
$F_0 = 0$				$F_0 = 0$			
1	0.0000	0.0726	0.2937	1	0.0000	0.0633	0.2843
2	0.0000	0.0043	0.0607	2	0.0000	0.0032	0.0496
$F_0 = +1$				$F_0 = +1$			
1	0.0000	0.0138	0.1203	1	0.0000	0.0098	0.1072
2	0.0000	0.0005	0.0150	2	0.0000	0.0002	0.0105

Table 4: Tranche hitting probabilities.

Hitting probabilities are lowest for case 1 (fixed LGD) and highest for case 4 (common factor) for any level of F_0. Despite our moderate parameterization[5] this result clearly reveals the systematic impact of LGD volatility and factor dependence. The relative differences between fixed and stochastic LGD cases are significant. Case 5 and 6 lie between case 1 and case 4 which is also a plausible result. Case 4 involves the same PD and LGD factor process so that systematically high PDs imply high LGDs. This relationship is

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[4]The intuitive reason for this phenomenon is shrinking subordination.

[5]$\tilde{\sigma}$ and ω_1 are fixed at rather low levels.

weakened in case 5 and completely absent in case 6. To summarize, modeling asset pool LGD as a PD factor related process significantly changes tranche risk. Moreover, an autocorrelated LGD process for its own is sufficient to increase tranche risk. The latter do not need to be conditionally related, yet, unconditionally. However, unconditional correlation, arising through similar initial values, is a realistic assumption.

3.2 Sensitivity of Tranche LGD

Besides tranche PD the other part of the tranche distribution is tranche LGD and this distribution is sensitive to systematic risk, too. We assume dependence of tranche LGD on asset pool LGD at first sight. However, as tranche LGD depends on cumulative asset pool loss we have also good reason to check for PD factor dependence (i.e., F_0 and F_t).

3.2.1 Initial Factor Dependence

In Table 5 mean tranche LGDs conditional upon F_0 for case 1 is printed.

Tranche	$t = 1$	$t = 3$	$t = 5$
$F_0 = -1$			
1	0.2487	0.4780	0.6426
2	—	0.4293	0.5642
$F_0 = 0$			
1	0.1667	0.3982	0.5486
2	—	0.3822	0.4996
$F_0 = +1$			
1	—	0.3349	0.4694
2	—	0.2879	0.4372

Table 5: Expected tranche LGD (Case 1).

We observe clear dependencies: higher levels of F_0 implicate lower average tranche LGDs. An explanation for this result is again the fact that L_t^{tr} is a figure that is based on cumulative pool loss which in turn is highly dependent on F_0 as we know from above.

3.2.2 Current Factor Dependence

Dependence on F_0 suggests dependence on F_t and the results provide evidence for this. Consider Figure 2. In accordance to tranche PD we obtain sensitivity diagrams similar in shape. Tranche LGD depends on F_t: average tranche LGDs grow as F_t decreases. Note that the curves do not approach zero at the right-hand side which depends on tranche width and asset pool LGD mean and volatility[6].

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[6]Expected LGD conditional upon F_t is useful when integrating a CDO in a superportfolio (e.g., the total bank portfolio).
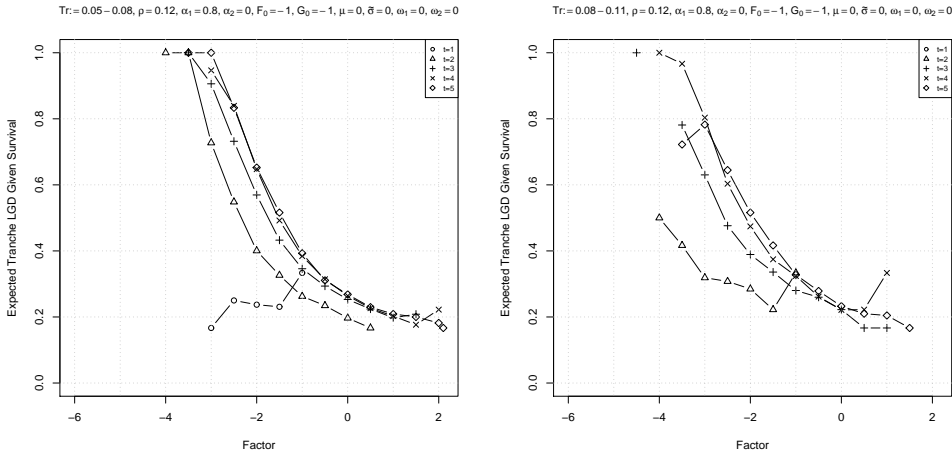
Figure 2: Factor sensitivity of mean tranche **LGD** (conditional upon survival) for tranches Tr_1 and Tr_2 ($F_0 = -1$).

3.2.3 Asset Pool LGD Dependence

Finally, similar to hitting probability, tranche LGD may also depend on asset pool LGD. Comparing our four different LGD models (case 1, 4, 5, and 6) we find no clear relationship (see Table 6).

Obviously, our LGD models imply lower tranche LGDs at maturity for tranche 1 and 2. We added an additional tranche $(A, B) = (0.11, 0.17)$ in order to show that higher tranches may be affected distinctly. For example, for this third tranche we observe that expected tranche LGD is higher in case 4 than in case 1 while it is lower for tranche 1 and 2.

3.3 Asset Pool LGD Shock

As a conclusion of our simulation study, we show the effects of an external (i.e., not model-implied) shock of asset pool LGDs. Such kind of shock could be observed only recently within the context of the house price crash in the U.S. For illustration assume that average LGD increases from 0.5 to 0.6. Table 7 shows the consequences in terms tranche PD and LGD.

The LGD stress implies soaring hitting probabilities, in some cases even three-times as high. Although not very surprising, this result underlines the amplified sensitivity of cumulative random variables with systematic dependence.

4 Conclusion

The recent financial turmoil has clearly shown that many market players had been completely ignorant of their risk exposition, largely because struc-

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	Case 1				Case 4		
Tranche	$t=1$	$t=3$	$t=5$	Tranche	$t=1$	$t=3$	$t=5$
$F_0=-1$				$F_0=-1$			
1	0.2459	0.4813	0.6451	1	0.1828	0.4676	0.6379
2	—	0.4079	0.5618	2	0.1648	0.4122	0.5613
3	—	0.1851	0.2922	3	—	0.2055	0.3193
$F_0=0$				$F_0=0$			
1	—	0.3978	0.5458	1	0.1721	0.3604	0.5221
2	—	0.3399	0.4957	2	—	0.3288	0.4796
3	—	0.1667	0.2551	3	—	0.1648	0.2777
$F_0=+1$				$F_0=+1$			
1	—	0.3433	0.4749	1	—	0.2876	0.4389
2	—	0.2619	0.4572	2	—	0.2997	0.4307
3	—		0.2381	3	—	0.0491	0.2439
	Case 5				Case 6		
Tranche	$t=1$	$t=3$	$t=5$	Tranche	$t=1$	$t=3$	$t=5$
$F_0=-1$				$F_0=-1$			
1	0.1738	0.4533	0.6312	1	0.1607	0.4294	0.6112
2	—	0.4001	0.5539	2	—	0.3724	0.5183
3	—	0.2026	0.3070	3	—	0.1896	0.2844
$F_0=0$				$F_0=0$			
1	0.1149	0.3494	0.5174	1	0.1008	0.3211	0.4880
2	—	0.3441	0.4689	2	—	0.3125	0.4346
3	—	0.1343	0.2528	3	—	0.1666	0.2406
$F_0=+1$				$F_0=+1$			
1	—	0.2830	0.4260	1	—	0.2512	0.3921
2	—	0.2989	0.4131	2	—	0.1874	0.3635
3	—	0.1606	0.2314	3	—	—	0.1715

Table 6: Expected Tranche Loss.

	Case 1 (without shock)				Case 1 (with shock)		
Tranche	$t=1$	$t=3$	$t=5$	Tranche	$t=1$	$t=3$	$t=5$
$F_0=-1$				$F_0=-1$			
1	0.0006	0.1807	0.4859	1	0.0047	0.3410	0.6600
2	0.0000	0.0165	0.1309	2	0.0000	0.0606	0.2702
$F_0=0$				$F_0=0$			
1	0.0000	0.0472	0.2473	1	0.0003	0.1227	0.4075
2	0.0000	0.0020	0.0399	2	0.0000	0.0099	0.1062
$F_0=+1$				$F_0=+1$			
1	0.0000	0.0078	0.0988	1	0.0000	0.0293	0.1975
2	0.0000	0.0002	0.0100	2	0.0000	0.0012	0.0324

Table 7: Tranche hitting probability.

tured finance products are different from standard securities. In this article, we provide fundamental tools for a sound risk assessment of structured finance products. We show that a scalar risk estimate is insufficient. Instead, risk measures have to be associated with scenarios and their probability of occurence. Given the relevance of scenarios, i.e., specific realisations of the systematic factor, we suggest a dynamic view in terms of both PD and LGD. Why? Structured finance products are based on a "diversified" loss random variable and are therefore much more sensitive to systematic risk than convential bonds. High systematic dependence implies that changes of

the systematic factor may have large impact. Through-the-cycle measures as commonly used by rating agencies for bonds do not reflect this. In other words, the economic cycle must not be disregarded with structured finance products. That is why we plead for dynamic stress tests which are certainly all the more relevant the more systematic risk sensitive a security is[7]. Above, we presented the necessary foundations.

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[7]Recent analyses of risk management causes of the credit crisis (e.g. SSG, 2008) confirm our position.

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Appendices

A   Default Threshold Calculation

A.1   Intertemporal Unconditional Covariance

$Cov\left(R_{it}, R_{it'}\right) = \mathbb{E}[R_{it}, R_{it'}] - \mathbb{E}[R_{it}]\mathbb{E}[R_{it'}]$

$$
\begin{aligned}
&= \rho\sigma^2 \sum_{j=0}^{t'-1} \alpha_1^{j+j+(t-t')}\mathbb{E}\left[W_{t-j}^2\right] + \rho\alpha_1^{t+t'}\mathbb{E}\left[F_0^2\right] + (1-\rho)^{\mathbf{1}_{\{t=t'\}}} \\
&= \rho\sigma^2 \sum_{j=0}^{t'-1} \alpha_1^{2j+(t-t')} + \rho\alpha_1^{t+t'} + (1-\rho)^{\mathbf{1}_{\{t=t'\}}} \\
&= \rho\left(1-\alpha_1^2\right) \sum_{j=0}^{t'-1} \alpha_1^{2j+(t-t')} + \rho\alpha_1^{t+t'} + (1-\rho)^{\mathbf{1}_{\{t=t'\}}} \\
&= \rho\left(\sum_{j=0}^{t'} \alpha_1^{2j+(t-t')} - \sum_{j=1}^{t'} \alpha_1^{2j+(t-t')}\right) + (1-\rho)^{\mathbf{1}_{\{t=t'\}}} \\
&= \rho\alpha_1^{|t-t'|} + (1-\rho)^{\mathbf{1}_{\{t=t'\}}}
\end{aligned}
$$

$$(34)$$

A.2 Intertemporal and Inter-Name Conditional Covariance

$$Cov\left(R_{it}, R_{jt'}\right) = \mathbb{E}\left[R_{it}, R_{jt'}\right] - \mathbb{E}[R_{it}]\mathbb{E}\left[R_{jt'}\right]$$

$$= \rho\sigma^2 \sum_{k=0}^{t'-1} \alpha^{k+k+(t-t')}\mathbb{E}\left[W_{t-k}^2\right] + \rho \cdot 1 \cdot \alpha_1^{t'+t-2} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$= \rho\sigma^2 \sum_{k=0}^{t'-2} \alpha^{2k+(t-t')} + \rho \cdot 1 \cdot \alpha_1^{t'+t-2} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$= \rho\left(1-\alpha_1^2\right) \sum_{k=0}^{t'-2} \alpha_1^{2k+(t-t')} + \rho \cdot 1 \cdot \alpha_1^{t'+t-2} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$= \rho\left(\sum_{k=0}^{t'-2} \alpha_1^{2k+(t-t')} - \sum_{k=1}^{t'-1} \alpha_1^{2k+(t-t')}\right) + \rho \cdot 1 \cdot \alpha_1^{t'+t-2} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$= \rho\alpha_1^{|t-t'|} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$(35)$$

A.3 Intertemporal and Inter-Name Conditional Covariance, Normalized

$$Cov\left(R_{it}, R_{jt'}\right) = \mathbb{E}\left[R_{it}, R_{jt'}\right] - \mathbb{E}[R_{it}]\mathbb{E}\left[R_{jt'}\right]$$

$$= \rho\sigma^2 \sum_{k=0}^{t'-1} \alpha^{k+k+(t-t')}\mathbb{E}\left[W_{t-k}^2\right] + \rho\alpha^{t+t'}f_0^2 + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}} - \rho\alpha^{t+t'}f_0^2$$

$$= \rho\sigma^2 \sum_{k=0}^{t'-1} \alpha^{2k+(t-t')} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$= \rho\left(1-\alpha_1^2\right) \sum_{k=0}^{t'-1} \alpha_1^{2k+(t-t')} + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$= \rho\left(\sum_{k=0}^{t'-1} \alpha_1^{2k+(t-t')} - \sum_{k=1}^{t'} \alpha_1^{2k+(t-t')}\right) + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$= \rho\left(\alpha_1^{|t-t'|} - \alpha_1^{|t+t'|}\right) + (1-\rho)^{\mathbf{1}_{\{t=t',i=j\}}}$$

$$(36)$$

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