

Is Diversification Possible with CDOs?  
Promises and Fallacies of an Investment Class

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# Is Diversification Possible with CDOs? Promises and Fallacies of an Investment Class

## Abstract

Diversification has been a frequently stated benefit of structured securitizations. However, in the course of the subprime credit crisis CDOs and especially ABS CDOs turned out to be risk concentrations. In this article, we show that pool derivatives indeed imply diversification, however, not necessarily for the investor's benefit. We examine the risk profiles of CDO tranches as well as their risk contributions to investor portfolios. We find that irrespective of any pool diversification the resulting tranches have always significantly higher risk concentrations than comparable bonds. Finally, based on the analysis of common structuring patterns, we conclude that it is much more difficult to achieve the same level of risk in a tranche portfolio as in a bond portfolio, even if the collateral universe is much larger than the corporate or sovereign bond universe.

KEYWORDS: *Diversification; CDO; Subprime Crisis*

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# 1 Introduction

The enormous growth of structured finance markets is due to a series of seemingly obvious benefits of products like CDOs (e.g. Kiff et al. 2004, Partnoy & Skeel 2007, Bluhm & Overbeck 2007).

On the one hand, financial institutions were attracted by (1) arbitrage opportunities and commission income, (2) diversification potentials and (3) more profitable risk allocations. Investors, on the other hand, (1) found adequate supply for their rating-specific demand, (2) could earn higher spreads than with conventional bonds, and (3) finally were pleased to be invested in ostensibly highly diversified collateral pools. Finally, regulators saw in securitization the potential of greater spread of risks and thus more systemic stability. For example, in an early 2007 report Kiff & Mills (2007) state that despite looming subprime problems “[...] [t]he dispersion of credit risk to a broader group of investors has nevertheless helped to make the U.S. financial system more resilient.” With hindsight this claim turns out to be wrong.

The financial crisis brought home to us that some of these promises were temporarily or permanently false. One pending issue is the appropriate pricing of CDOs as compared with other asset markets and as measured by systematic risk (Eckner 2008, Coval et al. 2009*b,a*, Brennan et al. 2009). Another open issue is diversification. To cite Cohen & Remolona (2008): “[...] it is still puzzling how instruments that were designed to spread and diversify risks ended up concentrating the risks.”

Investment banks, investors, and regulators have different views on the diversification benefits of CDOs. Arrangers of CDOs try to diversify their collateral pools by collecting assets from diverse sectors, countries or regions (McDermott 2001). We call this “factor diversification”. They also diversify risks by pooling several assets and thus reduce the magnitude of idiosyncratic risks. We call this “name diversification”. Banks seem to have two motivations for that. First, it was a value proposition when selling CDO tranches. Since the latter were deemed as “less risky” if the underlying pools were diversified a pool with higher, say, diversity score<sup>1</sup> (this is a measure of diversity created by Moody’s) was most likely easier to sell. Second, a pool with low concentration risk and high diversity score, respectively, has a loss distribution with a much lighter tail. This in turn implies a larger portion of senior liabilities and thus lower costs of funding. A prerequisite

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<sup>1</sup>The diversity score is to adapt a binomial distribution to the tail shape of a Vasicek distribution with corresponding correlation. The higher the concentration, the lower the score. For example, if in a pool with ten assets all are from the same issuer then the diversity score equals unity (highest score, lowest diversification). If all come from different issuers but from the same sector then the diversity score equals 3.98. Finally, if all ten assets are from different sectors, then the diversity score equals 10. The diversity score is used in Moody’s Binomial Expansion Technique (BET) as well as for Correlated BET. In the former case, the score is determined based on industry classification. The latter is developed for more complex assets and relies on an explicit correlation assumption.

for this, however, is a good rating (e.g., triple A) by one of the credit rating agencies (CRA). CRAs include concentration risk measures in their valuation models. For instance, in their Binomial Expansion Technique (BET) Moody's use the diversity score. In general, concentrations are included via a hierarchical (e.g., intra-industry and inter-industry) correlation structure.

Investors, by contrast, seem to have a different view on diversification. On the one hand, it appears that if the pool backing a CDO tranche is diversified the tranche itself is "diversified" (i.e., less risky), too. That is, if the pool is diversified, the resulting tranches contribute less to any concentration risks in the investor portfolio. To cite from a report of Nomura (2004): "[...] A CDO sponsor tries to create value by assembling a well-diversified portfolio of assets to back its CDO. In principle, diversification within a CDO's portfolio can make it stronger than merely the sum of its parts. [...]" On the other hand, investors see a diversification benefit of CDOs in the wealth of possible collateral types. Indeed, the universe of claims that can be securitized appears much larger than, that of corporate bonds or treasury bills. Thus, it appears beneficial to invest in different tranches backed by collateral pools with as little correlation as possible.

Finally, regulatory authorities see the diversification benefits of securitizations in a reduction of systemic risk. For example, a recent report of the International Monetary Fund (IMF 2008) states that "[...] Structured finance can be beneficial, allowing risks to be spread across a larger group of investors, [...]" In other words, securitization admits new investor groups to gain exposure to previously inaccessible asset classes. As a result, risk concentrations on a small subset of institutions decreases and the stability of the financial system increases.

There have been hardly any scientific contributions on diversification benefits and concentration dangers associated with CDOs in the past. Only gradually, as a reaction to the subprime crisis, more comments can be found pointing to the issue (Fitch 2008, 2009, Donhauser et al. 2010). A related topic which is also discussed only poorly in the literature until the beginning of the crisis is the specific systematic risk sensitivity of structured credit products. Per construction, CDO tranches have leveraged systematic risks (e.g. Donhauser et al. 2010, Jobst 2005, IMF 2008, Coval et al. 2009a, Brennan et al. 2009, Hu 2007) which in turn implies concentration risks in a portfolio.

One of the big topics in the structured credit literature associated with concentration risk was correlation. However, it was mainly the market-implied correlations that were of interest, not fundamental correlations. These market correlations were used as indicators of the market's expectations concerning default clustering. They are only available for liquidly traded derivatives like credit index tranches. Most cash-flow structured credit products, however, are secondary market products where no market prices do exist.

To summarize, our article gives answers to the following pressing questions:

1. How does asset pool composition (e.g., with ABS or ABS CDOs) translate into liabilities with predefined ratings?
2. How successful is diversification in view of the resulting tranches' stand-alone risk measures?
3. How do bond portfolios and tranche portfolios differ in terms of risk?

The rest of the article is organized as follows. In the next section we first outline a multi-factor collateral pool and tranche loss model and provide a variance-based measure of systematic risk which is applicable in this setting. Then, in Section 3, we fix a rating structure for the liabilities of the CDO and study the effect of various diversification actions on (1) the size of the resulting tranches and (2) relevant risk measures of the tranches. We draw comparisons with conventional fixed-income assets like bonds or pass-through securities. In Section 4 the risk and diversification benefits of bond portfolios and tranche portfolios are compared from an investor's point of view. In a concluding section, we relate our findings to phenomena of the subprime credit crisis.

## 2 Systematic Risk, Concentration Risk, and Diversification in Multi-Factor Model

In this section we define a CDO collateral pool and tranche loss model. Furthermore, we define statistics to measure and compare the systematic risk inherent in the tranches. In a single factor model the factor loading can be directly used as a measure. Not so, however, when multiple factors do exist.

### 2.1 Asset Pool and Tranche Loss Model

The asset pool comprises  $I$  defaultable securities. The creditworthiness of each of these loans, bonds or tranches  $i \in \{1, \dots, I\}$  is represented by an asset value variable

$$R_i = \sqrt{\rho_i}M_j + \sqrt{1 - \rho_i}U_i$$

where  $M_j, j \in \{1, \dots, J\}$ , is a systematic risk factor and  $U_i$  is an idiosyncratic (i.e., obligor-specific) factor. Both are standard normal. Each latent factor  $M_j$  depends on a common background factor  $M$

$$M_j = \sqrt{\omega_j}M + \sqrt{1 - \omega_j}W_j$$

This establishes dependencies among the systematic factors.

Usually, some names in the pool load on the same factor  $M_j$ . Thus, the pool contains  $n$  non-overlapping homogeneous subgroups  $i_j \subseteq \{1, \dots, I\}$ , i.e., any obligor belongs to exactly one subgroup  $j$  and  $i_j$  denotes the index set of obligors belonging to subgroup  $j$ . All obligors in the same subgroup load on the same factor  $M_j$ .

Now given these asset value variables, obligor  $i$ 's default is defined as the event that  $R_i$  falls short of a threshold  $c_i$

$$R_i < c_i \tag{1}$$

which is calibrated to imply the obligor's PD. That is,

$$\lambda_i = \mathbb{P}[R_i < c_i]$$

and

$$c_i = \Phi^{-1}(\lambda_i)$$

The portfolio loss is the weighted sum of these random default indicators, i.e.,

$$\begin{aligned} L &= \sum_i L_i \\ &= \sum_i w_i \mathbf{1}_{\{R_i < c_i\}} \text{LGD}_i \end{aligned} \tag{2}$$

where  $w_i$  denotes the notional weight of asset  $i$ ,  $\mathbf{1}_{\{\cdot\}}$  is an indicator function, and  $\text{LGD}_i$  is obligor  $i$ 's loss given default.

The liability part of a CDO is structured into horizontal loss tranches. "Horizontal" means of different seniority. The loss of tranche  $0 \leq A^{(\text{tr})} < B^{(\text{tr})} \leq 1$  is given by

$$L^{(\text{tr})} = \frac{\min(B^{(\text{tr})}, L) - \min(A^{(\text{tr})}, L)}{B^{(\text{tr})} - A^{(\text{tr})}} \tag{3}$$

Thus, for  $L \leq A^{(\text{tr})}$  tranche tr does not incur losses and it is completely lost for  $L \geq B^{(\text{tr})}$ .

Furthermore, the opposite of a horizontal slice (tranche) is a vertical slice of the portfolio. Vertical slices offer no seniority differences, i.e., each slice obtains its notional share of the total portfolio loss  $L$ . We refer to such vertical slices as "portfolio investments" or "pass-through securities".

## 2.2 Concentration Risk and Diversification

In the Basel II capital convergence rules (BIS 2006, §770) concentration risk is defined as "[...] any single exposure or group of exposures with the potential to produce losses large enough (relative to bank's capital, total assets

or overall risk level) to threaten a bank’s health or ability to maintain its core operations.” In §772 it is further related to risk factors: “Credit risk concentrations, by their nature, are based on common or correlated risk factors, which, in times of stress, have an adverse effect on the creditworthiness of each of the individual counterparties making up the concentration.”

Thus, default of large portfolio fractions is more likely in the presence of

1. Name Concentration
2. Factor Concentration
  - (a) “between factors”
  - (b) “within factors”

Name concentration or exposure concentration refers to portfolio positions with large exposure so that their default represents a large fraction of the whole portfolio. Factor concentration is characterized by high potential of large losses due to either simultaneous default of many pool positions or simultaneous default of obligors with high LGDs. These collective effects are caused by common factors which influence the default behavior of many obligors simultaneously and similarly. Factors often differ in terms of region or industry. Another difference has to be made between factor load and factor dependence. On the one hand, there may be concentrations on certain factors, i.e., major proportions of the loss mass<sup>2</sup> is concentrated on some factors (e.g., when 90% of the portfolio positions are related to the automobile sector we have a case of sector concentration). On the other hand, factor concentrations may also exist in terms of stress concentration, i.e., how strong the portfolio loss mass is restricted to adverse factor scenarios. For example, we shall see below that structured finance products are particularly stress scenario concentrated, i.e., in good times their default risk is almost zero while it may increase rapidly in very bad times.

Now, reducing name or factor concentrations is called “diversification”. By putting the exposures of a portfolio on diverse risky obligors or factors reduces the total risk.

In the literature there is sometimes another type of concentration risk called “contagion risk” which refers to default or loss cascades among obligors, i.e., default of one obligor causes another to default. In a factor representation this means that one obligor is a risk factor to another and thus we consider this as a special case of factor concentration.

In order to measure concentration risk within the context of CDOs a statistic with two properties is necessary. First, it needs to be interpretable as a stand-alone measure so that our results are more generalizable. Stand-alone risk measures should indicate the potential concentration risk of an

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<sup>2</sup>We define “loss mass” as the probability mass of  $\mathbb{E}[L | (M_j)_{j=1}^J]$ .



asset instead of its concrete risk contribution within a specific portfolio. Second, because of mass singularities with CDO loss distributions we seek a non-quantile-based measure<sup>3</sup>.

We use the classical variance here since it is easy to calculate and does not require fixing a quantile. Furthermore, the variance has a well-known decomposition in systematic and unsystematic parts (“law of total variance”):

$$\mathbb{V}[L] = \underbrace{\mathbb{E}[\mathbb{V}[L | (M_j)_{j=1}^J]]}_{\text{mean unsystematic variance}} + \underbrace{\mathbb{V}[\mathbb{E}[L | (M_j)_{j=1}^J]]}_{\text{systematic variance}} \quad (4)$$

i.e., the mean variance about conditional expected loss (CEL) plus the variance of conditional expected loss.

We are interested in the second, the systematic component which is primarily responsible for the tails of the overall loss distribution. However, although  $\mathbb{V}[\mathbb{E}[L | (M_j)_{j=1}^J]]$  lends itself to compare loss variables with the same expectation, it is difficult to interpret in absolute terms and thus to compare with it investments with different expected loss. To that end, we relate it to the maximum variance, i.e., we define

$$\tilde{\sigma} = \frac{\mathbb{V}[\mathbb{E}[L | (M_j)_{j=1}^J]]}{\mathbb{V}[\mathbb{E}[L | (M_j)_{j=1}^J]_{\max}]} \quad (5)$$

where  $\mathbb{E}[L | (M_j)_{j=1}^J]_{\max}$  denotes a random variable relating to a conditional expected loss distribution with maximum variance, i.e., a two point distribution with all mass exclusively distributed on zero and the maximum possible loss of  $L$ , denoted by LGD. We call  $\tilde{\sigma}$  “normalized conditional expected loss (CEL) variance”. For the distribution of  $\mathbb{E}[L | (M_j)_{j=1}^J]_{\max}$  we impose the restriction that the two support end points are zero and LGD, respectively, and furthermore that

$$\mathbb{E}[\mathbb{E}[L | (M_j)_{j=1}^J]_{\max}] = \mathbb{E}[\mathbb{E}[L | (M_j)_{j=1}^J]] = \mathbb{E}[L] \quad (6)$$

Thus,

$$\mathbb{E}[\mathbb{E}[L | (M_j)_{j=1}^J]_{\max}] = \tilde{p} \cdot \text{LGD} = \mathbb{E}[L] \quad (7)$$

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<sup>3</sup>Quantile-based risk measures may be problematic with CDOs because of their typically possessing mass singularities at either of the support end points. As a result, all tranches except for the most senior one have a mass point at 100% which in turn makes tail risk measures (like value at risk, expected shortfall, etc.) insensitive at higher quantile levels.

where  $\tilde{p}$  denotes the probability mass on LGD. The rest of the mass,  $1 - \tilde{p}$ , is on zero. As a result, we obtain

$$\tilde{p} = \frac{\mathbb{E}[L]}{\text{LGD}} \quad (8)$$

Now the maximum variance can be calculated

$$\begin{aligned} \mathbb{V} \left[ \mathbb{E} \left[ L \mid (M_j)_{j=1}^J \right]_{\max} \right] &= 0 \cdot (1 - \tilde{p}) + \text{LGD}^2 \tilde{p} - \mathbb{E}[L]^2 \\ &= \text{LGD}^2 \cdot \frac{\mathbb{E}[L]}{\text{LGD}} - \mathbb{E}[L]^2 \\ &= \text{LGD} \cdot \mathbb{E}[L] - \mathbb{E}[L]^2 \end{aligned} \quad (9)$$

Our relative variance coefficient (5) approaches unity as  $\mathbb{E} \left[ L \mid (M_j)_{j=1}^J \right]$  converges to a “one-zero” jump function and it approaches zero the closer the profile to a straight line about expected loss. We show this for the special case of the Gaussian single risk factor model in Appendix B.

### 3 Concentration Risk and Diversification with ABS and CDOs

As outlined above, tranced securitizations like ABS or CDOs have been appreciated for their diversification potential. Nevertheless, the recent sub-prime debacle has showed clear patterns of concentration risk. This raises the following questions

1. How does asset pool composition (e.g., with ABS or ABS CDOs) translate into liabilities (subordination, width) with predefined ratings?
2. How successful is diversification in view of the resulting tranches' stand-alone risk measures?
3. How do bond portfolios and tranche portfolios differ in terms of risk?

We shall answer these questions now.

#### 3.1 Asset Pool Diversification Patterns

In a first step, we explain historically observed asset pool concentration and diversification patterns. Above, we distinguished name diversification and factor diversification. ABS asset pools are frequently highly name diversified. This is due to the size of asset pools which may be as high as several thousand names. However, there are also small asset pools, typically those of CDOs, with only one hundred or less names. We study both extremes by variation of the pool size,  $N$ .

In our model, systematic risk is due to common factors such as industrial or regional factors. The lower the overall load of the obligors in a pool on the same factor the lower their covariation. Now, an arranger has two major options to “diversify” risks. On the one hand, he may collect assets with high idiosyncratic loadings and low common factor loadings. On the other hand, he may collect assets loading on factors which are as independent as possible. In our model this corresponds to the levels of intra-factor and inter-factor correlation,  $\rho$  and  $\omega$ .

Asset managers commonly try to reduce systematic risks (e.g., increase the diversity score<sup>4</sup>) in order to get better ratings for the liabilities to be marketed (e.g. Nomura 2004, Lucas et al. 2006) or conversely to create a larger proportion of high grade assets. Asset pools with high systematic risk require much more credit enhancements for AAA tranches and are thus less attractive in arbitrage terms. Investors, on the other hand, are also interested in assets with low systematic risk or they require additional premium. This means that both parts seem to be interested in low levels of  $\rho$  and  $\omega$ . But in reality this target was often not achieved.

For example, during the golden years of subprime securitization (i.e., 2005 to early 2007) the issuance volume of ABS CDOs (i.e., CDOs with ABS collateral) grew considerably. Now, previous research has revealed that tranching always implies increased systematic risk sensitivity (e.g. Donhauser et al. 2010) which in turn implies that the collateral pools of ABS CDOs had high systematic risk sensitivity, i.e., the assets had high  $\rho$ . On the other hand, the assets of ABS CDOs were RMBS which were regionally diversified in the U.S. But obviously, this had no significant diversification effect. As a result, inter-factor correlation was high, too.

To summarize, in the past we have observed pools with high or low  $N$ , high or low  $\rho$ , and high or low  $\omega$ . Table 1 states examples of securitization asset pools with different degree of diversification.

		low	high
Intra-Sector Correlation	$\rho$	Credit Card ABS	ABS CDO
Inter-Sector Correlation	$\omega$	Multi-Sector CDO	U.S. Subprime RMBS
Portfolio Size	$N$	CDO	ABS

**Table 1:** Examples of securitization asset pools with different degree of diversification.

CDO asset pools are typically small ( $N = 100$ ) while ABS pools may become very large ( $N > 1000$ ). Corporate CDOs or multi-sector CDOs are backed by assets from diverse sectors while 2005-2007 U.S. subprime RMBS have homogeneous pools from a housing market with a strong com-

<sup>4</sup>The diversity score is a diversification measure developed and used by Moody’s.

mon factor<sup>5</sup>. Finally, credit card ABS or consumer ABS are less dependent from common risk factors while ABS CDO collateral is highly factor concentrated. Note that while it is known that RMBS pools are homogeneous and thus come from the same sector they seem to be deemed as having low intra-sector correlations which lowers the overall concentration risk. The same applies to ABS CDOs which have ABS collateral with high factor concentration but when the ABS come from diverse factors/sectors, the overall concentration is again low.

### 3.2 Configuration

To answer the above questions we proceed as follows. We choose an asset pool and fix a liability structure. The asset pool parameterization agrees with one of the patterns in Table 1. The liabilities are only defined in terms of their desired expected loss. The latter means that both tranche subordination and tranche width are not prespecified but have to be determined based on the expected loss requirements. Finally, we calculate stand-alone tranche risk measures. We repeat this for all high/low combinations of  $N$ ,  $\rho$ , and  $\omega$ , i.e., for eight asset pools altogether.

**Asset Pool** The asset pool is represented by the Gaussian factor model as outlined above with two systematic factors  $M_1$  and  $M_2$  having correlation  $\sqrt{\omega_1 \cdot \omega_2} = \omega$ . To simplify things, we assume that the asset pool is homogeneous in terms of PD, factor loading, LGD, and exposure weight, i.e.,  $\lambda_i = \lambda$ ,  $\rho_i = \rho$ ,  $\text{LGD}_i = \text{LGD}$ , and  $w_i = 1/N$ . We further assume that half of the obligors loads on  $M_1$  (i.e., belongs to subgroup  $i_1$ ) and the other half loads on  $M_2$  (i.e., belongs to subgroup  $i_2$ ).

Table 2 shows the set of pool setups which we are going to test. These are concrete values consistent with Table 1.

General Parameters	Symbol	low	high
Intra-Sector Correlation	$\rho$	0.20	0.75
Inter-Sector Correlation	$\omega$	0.05	1.00
Portfolio Size	N	100	1000

**Table 2:** Asset pool parameters.

$\rho = 0.20$  and  $\rho = 0.75$  are to reflect conventional and securitization claims, respectively. As extensively examined by Donhauser et al. (2010) and also by Moody's KMV (2008) representing a tranche as a loan in a single factor model requires much higher asset correlations. Thus, an asset

<sup>5</sup>Historically, U.S. housing markets were local. However, in the years before the crisis several factors have triggered a U.S. wide price increase.

correlation of  $\rho = 0.75$  is a typical level for mezzanine tranches<sup>6</sup>.  $\omega = 0.05$  and  $\omega = 1.00$  refer to asset pools with constituents coming from diverse and a single sector, respectively.  $N = 100$  and  $N = 1000$  admit studying name diversification. The two levels of PD and LGD correspond to 5-year Aaa and Baa Moody’s corporate bond ratings (Moody’s 2009).

**Capital Structure** The desired capital structure based on expected loss is given in Table 3. For the eight asset pool constellations we calculate a senior and a mezzanine tranche having these EL levels.

	Expected Loss
Senior (Aaa)	0.0002
Mezzanine (Baa)	0.0110

**Table 3:** Desired ratings of liabilities.

We took them from Moody’s (2009). All tranches are based on a pool with homogeneous PD  $\lambda = 0.0189$ . For  $\rho = 0.20$  the pool is assumed to be comprised of bonds, hence we use the bond LGD of Moody’s,  $\text{LGD} = 0.582$ . For  $\rho = 0.75$  the pool is assumed to be comprised of mezzanine tranches and hence we set LGD to unity. We use expected loss as a tranching criteria which is the official Moody’s criterion. Our tests of hitting probability as a tranching criterion brought no further insights and thus we do not report those results here.

We compare the resulting tranches with bonds and pass-through securities having the same EL. The latter are backed by a pool of bonds. The bond parameters are given in Table 4.

General Parameters	Symbol	Aaa	Baa
PD	$\lambda$	0.00088	0.0189
LGD	LGD	0.227	0.582
EL	EL	0.0002	0.0110

**Table 4:** Bond parameters.

### 3.3 Results

#### 3.3.1 Capital structure

The resulting capital structures are given in Table 5. We see that lowering  $\rho$  or  $\omega$ , i.e., factor diversification, enlarges the senior tranche and shortens

<sup>6</sup>Senior tranches have usually lower implied correlations because of their lower leverage.

the mezzanine tranche. When both  $\omega$  and  $\rho$  are very high the AAA senior proportion becomes extremely small. Such levels have never been observed in reality where the senior part usually accounts for the major part of the capital structure. By contrast, name diversification, i.e., increasing  $N$  has hardly any effect.

	$\omega = 0.05$				$\omega = 1.00$			
	$N = 100$		$N = 1000$		$N = 100$		$N = 1000$	
	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$
Equity	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Mezzanine	0.0530	0.2060	0.0470	0.2070	0.0670	0.1240	0.0630	0.1230
Senior	0.0570	0.4560	0.0480	0.4460	0.0880	0.9680	0.0800	0.9530

**Table 5:** Capital structure, attachment points. EL tranching.

### 3.3.2 Unexpected loss measures

Normalized CEL variances,  $\tilde{\sigma}$ , for all assets (tranches, bonds, portfolios) are given in Table 6. First of all, the mezzanine tranches in all eight scenarios have very high  $\tilde{\sigma}$ . The senior tranches achieve lower levels when both  $\rho$  and  $\omega$  are low and high levels when both asset pool correlations are high. Comparing tranches with their rating-analog bonds and pool investments tranches have clearly higher variance portions (and thus facto sensitivity) in all scenarios. Reducing  $\rho$ , i.e., intra-factor correlation, implies significantly lower  $\tilde{\sigma}$  with bond and portfolio investments. For instance, for  $\omega = 0.05$  when moving from  $\rho = 0.75$  to  $\rho = 0.20$   $\tilde{\sigma}$  declines by factor 10 to 40 with bonds and portfolios. By contrast, corresponding effects for mezzanine and senior tranches are clearly smaller (factor 7 and 1.5, respectively). In addition, while lower  $\rho$  implies lower  $\tilde{\sigma}$  with senior tranches the effect is unclear for mezzanine tranches.

Reducing  $\omega$ , i.e., inter-factor correlation, implies moderately lower  $\tilde{\sigma}$  with portfolio investments (factor 2). Bonds are of course unaffected. Senior tranche risks are also reduced by a factor of about two. Again, for mezzanine tranches the effect is small and unclear as for direction, i.e., in some cases rising, in some cases falling.

Finally, as for name diversification there is of course no effect with bond and portfolio investments. However, for tranches, we find that increasing the number of names always implies increased  $\tilde{\sigma}$ . Although of small order of magnitude, the effect is not in line with usual diversification arguments according to which we would expect lower tranche risks.

Finally, an explanatory note on tranche asset correlations. Above, we stated that tranches which are represented in the Gaussian single risk-factor bond model have always leveraged asset correlation in comparison with the

underlying pool. For example, when the pool has average asset correlation of  $\rho = 0.20$ , a mezzanine tranche represented as a bond has typically  $\rho$  between 0.70 and 0.90. We are able to show this now quite simply based on  $\tilde{\sigma}$ , our normalized systematic risk measure. In Table 7 we show asset correlations of bonds represented in a Gaussian single factor model having the same normalized systematic variance,  $\tilde{\sigma}$ , and the same expected loss as the securities in Table 6. We see that the implied correlations of the mezzanine tranche is greater than 80% in all eight cases. By contrast, a conventional bond has asset correlations between 5% and 20%. Furthermore, all tranche correlations are significantly higher than the corresponding bond or pool correlations. Altogether, this again shows that both their systematic risk sensitivity as well as their contribution to portfolio concentrations is significantly higher.

	$\omega = 0.05$				$\omega = 1.00$			
	$N = 100$		$N = 1000$		$N = 100$		$N = 1000$	
	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$
Equity	0.2041	0.5668	0.2398	0.5714	0.2716	0.6965	0.2944	0.7053
Mezzanine	0.4119	0.6247	0.7825	0.6595	0.5360	0.4771	0.8052	0.4888
Senior	0.0314	0.2130	0.0471	0.2184	0.0781	0.5945	0.0970	0.6835
Pool (EL = 0.0110)	0.0180	0.1770	0.0180	0.1769	0.0351	0.3498	0.0348	0.3493
Pool (EL = 0.0002)	0.0028	0.1011	0.0028	0.1010	0.0054	0.2019	0.0055	0.2017
Bond (EL = 0.0110)	0.0355	0.3485	0.0355	0.3485	0.0368	0.3507	0.0368	0.3507
Bond (EL = 0.0002)	0.0053	0.2008	0.0053	0.2008	0.0060	0.2148	0.0060	0.2148

**Table 6:** Normalized CEL variance,  $\tilde{\sigma}$ .

	$\omega = 0.05$				$\omega = 1.00$			
	$N = 100$		$N = 1000$		$N = 100$		$N = 1000$	
	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$	$\rho = 20\%$	$\rho = 75\%$
Equity	0.3641	0.8524	0.4084	0.8555	0.4891	0.9282	0.5144	0.9322
Mezzanine	0.8180	0.9346	0.9771	0.9467	0.8975	0.8665	0.9830	0.8730
Senior	0.4794	0.8043	0.5376	0.8087	0.6167	0.9614	0.6535	0.9772
Pool (EL = 0.0110)	0.1216	0.5453	0.1215	0.5454	0.2023	0.7512	0.2004	0.7511
Pool (EL = 0.0002)	0.1377	0.6183	0.1384	0.6181	0.1994	0.7520	0.2007	0.7513
Bond (EL = 0.0110)	0.2031	0.7500	0.2031	0.7500	0.2127	0.7536	0.2127	0.7536
Bond (EL = 0.0002)	0.1895	0.7496	0.1895	0.7496	0.2184	0.7664	0.2184	0.7664

**Table 7:** Asset correlation of a single-factor bond having normalized CEL variance  $\tilde{\sigma}$  as reported in Table 6.

## 4 Diversification Potentials of CDO and Corporate Bond Portfolios in Comparison

In the last section we have studied the effect of diversification activities within the context of typical tranching patterns. We found that the concentration risk of tranches is always increased as compared with bonds or horizontal pool investments. But even so the question arises whether these risk concentrations could be eliminated by investing in as many different factors as possible. In other words, although each individual tranche has high risk concentration an investor may wonder whether this is indeed relevant. He only needs to collect tranches based on asset pools driven by as many different factors (e.g., countries or sectors) as possible. Given that any collection of assets can be securitized, at least in theory, this seems to be a realistic claim.

In the previous section we used a two-factor model for the CDO collateral pools. The investor portfolios we model now are rather similar. The only difference is that we generalize from two to  $n$  factors. That is, we have a homogeneous portfolio of  $I$  fixed-income securities. Each of these bonds or tranches  $i$  has an associated asset value process

$$R_i = \sqrt{\rho}M_j + \sqrt{1-\rho}U_i$$

Each of these asset values is associated with a latent factor  $M_j$  which in turn depends on a common background factor  $M$

$$M_j = \sqrt{\omega}M + \sqrt{1-\omega}W_j$$

so that any two “systematic” factors are correlated

$$\text{Corr}(M_j, M_{j'}) = \omega$$

Substituting  $M_j$  in  $R_i$  yields

$$\begin{aligned} R_i &= \sqrt{\rho}(\sqrt{\omega}M + \sqrt{1-\omega}W_j) + \sqrt{1-\rho}U_i \\ &= \sqrt{\rho}(\sqrt{\omega}M + \sqrt{1-\omega}W_j) + \sqrt{1-\rho}U_i \\ &= \sqrt{\rho}\sqrt{\omega}M + \sqrt{\rho}\sqrt{1-\omega}W_j + \sqrt{1-\rho}U_i \\ &= \sqrt{\rho}\sqrt{\omega}M + \sqrt{1-\rho\omega}\tilde{U}_i \end{aligned} \tag{10}$$

Thus, if  $i, i' \in i_j$

$$\begin{aligned} R_i &= \sqrt{\rho}M_j + \sqrt{1-\rho}U_i \\ R_{i'} &= \sqrt{\rho}M_j + \sqrt{1-\rho}U_{i'} \end{aligned} \tag{11}$$

and  $\text{Corr}(R_i, R_{i'}) = \rho$ .



By contrast, if  $i, \in i_j, i' \in i_{j'}$  and  $j \neq j'$ ,

$$\begin{aligned}
R_i &= \sqrt{\rho}M_j + \sqrt{1-\rho}U_i \\
&= \sqrt{\rho}\sqrt{\omega}M + \sqrt{1-\rho\omega}\tilde{U}_i \\
R_{i'} &= \sqrt{\rho}M_{j'} + \sqrt{1-\rho}U_{i'} \\
&= \sqrt{\rho}\sqrt{\omega}M + \sqrt{1-\rho\omega}\tilde{U}_{i'}
\end{aligned} \tag{12}$$

and  $\text{Corr}(R_i, R_{i'}) = \rho \cdot \omega$ . Note that  $\tilde{U}_i$  are correlated for any two positions from the same subgroup, yet, they are uncorrelated for any two positions from different subgroups.

Thus, the portfolio is divided into  $n$  subgroups of assets each with an individual common risk factor. The portfolio may contain either bonds or tranches. Both asset classes differ in terms of their asset correlation  $\rho$ . For CDOs they are naturally significantly higher than for classical bonds. We used this fact already when studying collateral pools with assets (i.e., tranches) having correlations as high as 75%.

Furthermore, we assume equally-sized subgroups  $j$  so that each has portfolio share of  $1/n$ . Let  $L_j$  denote loss in subgroup  $j$ , then  $L_j/n$  is the loss share of subgroup  $j$  in the total portfolio. For the total portfolio variance we obtain

$$\begin{aligned}
\mathbb{V}[L] &= \mathbb{V}\left[\sum_j \frac{1}{n}L_j\right] \\
&= \sum_j \frac{1}{n^2}\mathbb{V}[L_j] + \sum_{j < j'} 2 \cdot \frac{1}{n^2}\text{Cov}(L_j, L_{j'})
\end{aligned} \tag{13}$$

Now the subgroup variances are given by

$$\begin{aligned}
\mathbb{V}[L_j] &= \mathbb{V}[\mathbb{E}[L_j | M]] + \mathbb{E}[\mathbb{V}[L_j | M]] \\
&= \text{LGD}^2 \left[ \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \lambda^2 + \frac{1}{N} (\lambda - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho)) \right]
\end{aligned} \tag{14}$$

where  $N$  denotes the absolute subgroup size, i.e., each subgroup contains  $N$  obligors.

The inter-subgroup covariation is given by

$$\begin{aligned}
\text{Cov}(L_j, L_{j'}) &= \mathbb{E}[\mathbb{E}[L_j \cdot L_{j'} | M]] - \mathbb{E}[L_j]\mathbb{E}[L_{j'}] \\
&= \text{LGD}^2 \cdot \mathbb{E}[\lambda_j(M)\lambda_{j'}(M)] - \lambda^2 \cdot \text{LGD}^2 \\
&= \text{LGD}^2 \cdot \mathbb{E}[\mathbb{P}[R_j < c_j, R_{j'} < c_{j'} | M]] - \lambda^2 \cdot \text{LGD}^2 \\
&= \text{LGD}^2 \cdot \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho\omega) - \lambda^2 \cdot \text{LGD}^2
\end{aligned} \tag{15}$$

Note that the last line of (15) looks similar to the first two terms of (14). Indeed, it is the same as long as  $\omega = 1$  which represents the case of all loading on the same factor  $M_j$ .

Also note where the correlation of the bivariate normal comes from. It is the correlation between any  $R_i$  and  $R_{i'}$  where  $i \in i_j$  and  $i' \in i_{j'}$  and  $j \neq j'$ :

$$\text{Corr}(R_i, R_{i'}) = \sqrt{\rho\omega}\sqrt{\rho\omega} = \rho\omega$$

Substituting (14) and (15) into (13) we obtain

$$\begin{aligned} \mathbb{V}[L] &= \mathbb{V}\left[\sum_j \frac{1}{n} L_j\right] \\ &= \sum_j \frac{1}{n^2} \mathbb{V}[L_j] + \sum_{j < j'} 2 \cdot \frac{1}{n^2} \text{Cov}(L_j, L_{j'}) \\ &= \text{LGD}^2 \cdot [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho\omega) - \lambda^2] \\ &\quad + \text{LGD}^2 \cdot \frac{1}{n} \left[ \frac{1}{N} (\lambda - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho)) \right] \\ &\quad + \text{LGD}^2 \cdot \frac{1}{n} \cdot [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho\omega)] \end{aligned} \tag{16}$$

The variance clearly decreases as  $n$ , the number of (equally weighted) factors in the portfolio, increases. However, this effect becomes less significant as  $N$  increases and also as  $\omega$  approaches unity. The  $N$  effect is simple name diversification. The second summand tends to zero as  $N$  increases. The  $\omega$  effect means that as  $\omega \rightarrow 1$  the difference in the third line (third summand) tends to zero. Clearly, the diversification effect of investments in a larger number of factors  $n$  is less pronounced the higher the inter-factor correlation  $\omega$ . Finally note that the first summand cannot be decreased by increasing  $n$ . In other words, this baseline risk is not diversifiable by investment in more subsectors. The only way to reduce it is by lowering  $\omega$  and  $\rho$ .

#### 4.1 Analysis

To analyze the model we need to set some real-world restrictions. We do have quite reliable estimates of the factor loadings  $\rho$  of bonds and tranches. For bonds  $0.05 \leq \rho \leq 0.20$  is quite realistic. For tranches, much higher factor loadings are necessary, usually in the range  $0.60 \leq \rho \leq 0.95$ . Our knowledge on how many factors are available and how high they correlate, however, is rather limited. The general assumption is that the corporate bond universe is clearly smaller than the world of securitizations. Hence,  $n$  and  $\omega$  need to be our key sensitivity parameters.

Now we perform the following analysis. We compare two portfolios, a bond portfolio and a tranche portfolio. Both differ only in terms of  $n$ , the number of factors which the assets load on. All else  $(\lambda, \text{LGD}, \rho, \omega)$  is identical. We choose a level of  $\omega$  between zero and unity. We also choose the number of factors  $n_B$  in the bond portfolio. Then we ask which number of factors  $n_T$  are necessary in the tranche portfolio in order to obtain at least the bond portfolio variance. Thus, we require that bond and tranche portfolio variance be equal:

$$\mathbb{V}[L_B] = \mathbb{V}[L_T] \quad (17)$$

Upon substitution of (16) and rearrangement we obtain

$$n_T = n_B \cdot \frac{\text{LGD}_T^2}{\text{LGD}_B^2} \cdot \frac{\zeta_1}{\zeta_2} \quad (18)$$

where

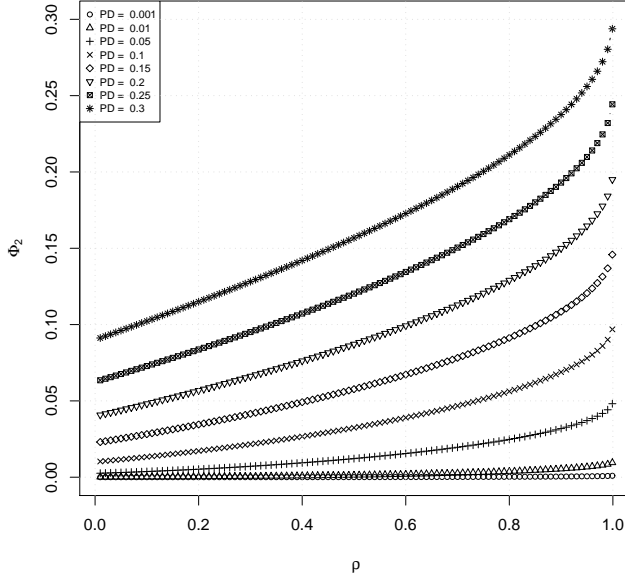
$$\begin{aligned} \zeta_1 &= \left[ \frac{1}{N} (\lambda - \Phi_2(\cdot; \rho_T)) \right] \\ &\quad + [\Phi_2(\cdot; \rho_T) - \Phi_2(\cdot; \rho_T \omega)] \\ \zeta_2 &= \left[ \frac{1}{N} (\lambda - \Phi_2(\cdot; \rho_B)) \right] \\ &\quad + [\Phi_2(\cdot; \rho_B) - \Phi_2(\cdot; \rho_B \omega)] \\ &\quad + n_B \left[ [\Phi_2(\cdot; \rho_B \omega) - \lambda^2] - \frac{\text{LGD}_T^2}{\text{LGD}_B^2} [\Phi_2(\cdot; \rho_T \omega) - \lambda^2] \right] \end{aligned} \quad (19)$$

If  $\text{LGD}_B = \text{LGD}_T$  we obtain the following bond-tranche-portfolio relationship:

$$n_T = n_B \cdot \frac{[\frac{1}{N} (\lambda - \Phi_2(\cdot; \rho_T))] + [\Phi_2(\cdot; \rho_T) - \Phi_2(\cdot; \rho_T \omega)]}{[\frac{1}{N} (\lambda - \Phi_2(\cdot; \rho_B))] + [\Phi_2(\cdot; \rho_B) - \Phi_2(\cdot; \rho_B \omega)] + n_B [\Phi_2(\cdot; \rho_B \omega) - \Phi_2(\cdot; \rho_T \omega)]} \quad (20)$$

This formula relates the number of factors in a tranche portfolio to the number of factors in a bond portfolio so that both have equal variance. If the quotient on the right hand side (we call this ‘‘bond-tranche ratio’’) is greater than unity, then  $n_T > n_B$ .

First, since  $\rho_T > \rho_B$  the first term in the numerator (the idiosyncratic term) is smaller than its corresponding term in the denominator. Thus, larger subgroups (implying a reduction of idiosyncratic risk) increase the bond-tranche ratio. Second, regarding the second terms the numerator term is also larger. The reason is that  $\Phi_2(\cdot; \rho_T) > \Phi_2(\cdot; \rho_B)$  and  $\Phi_2(\cdot; \rho_T \omega) > \Phi_2(\cdot; \rho_B \omega)$ , it is true, but we know that  $\Phi_2(\cdot; \rho)$  is convex,



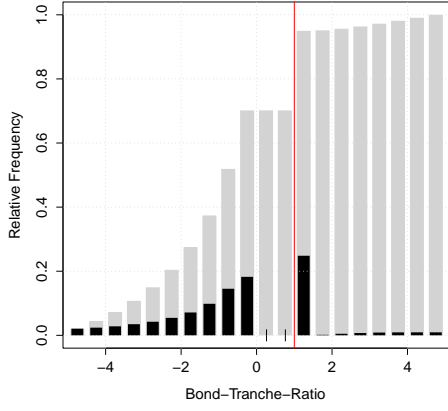
**Figure 1:** Bivariate Gaussian Distribution  $\Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho)$ .

i.e., its slope rises with  $\rho$ . Figure 1 shows this very clearly. Finally, the third term in the denominator. The difference  $\Phi_2(\cdot; \rho_B \omega) - \Phi_2(\cdot; \rho_T \omega)$  is negative because  $\rho_T > \rho_B$ . Thus, a larger number of factors in the bond portfolio reduce the denominator and increase the quotient. Through this term it is possible that the denominator becomes negative. In that case, there is no positive solution for  $n_T$ , i.e., there is no increase in the number of factors in the tranche portfolio that drives the portfolio variance down to the bond portfolio variance level. Finally, from all this we conclude that  $n_T > n_B$ .

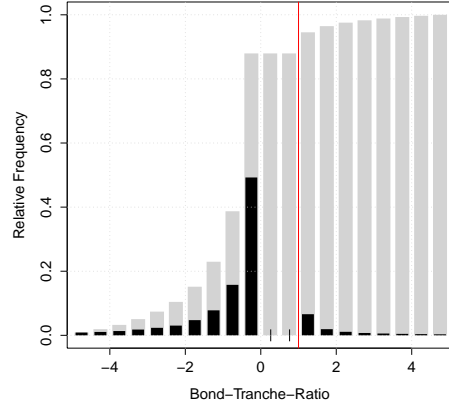
We want to collect some numerical facts now. To that end, we perform the following simulation study. We choose different combinations of  $\rho_T, \rho_B, \omega, \lambda, N$ , and  $n_B$  from the range of plausible levels (see Table 8) and calculate the bond-tranche-ratio.

Our results are shown in Figure 2.

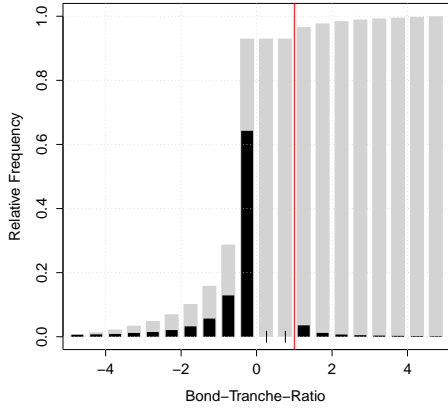
The graphs show relative frequencies as well as cumulative relative frequencies of the bond-tranche-ratio. The subgraphs refer to different levels of  $n_B$ . First of all, we observe that there are no observations between zero and unity. This confirms our reasoning from above. Second, we see that in most cases the major mass of observations is in the negative region which means that there is no positive  $n_T$  allowing to achieve variances as low as in the bond portfolio. Note that as  $n_B$  rises, more and more mass is be-



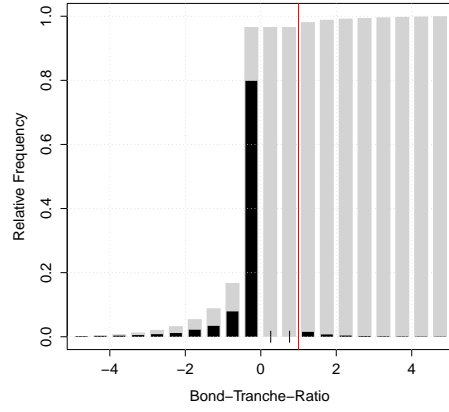
(a)  $n_B = 1$



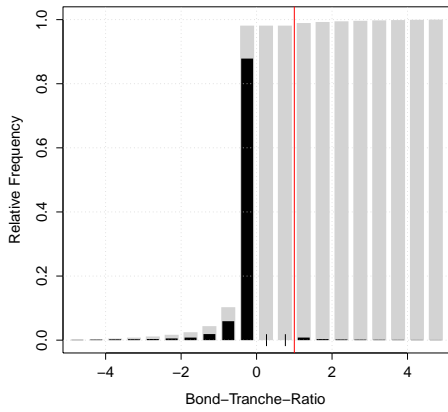
(b)  $n_B = 5$



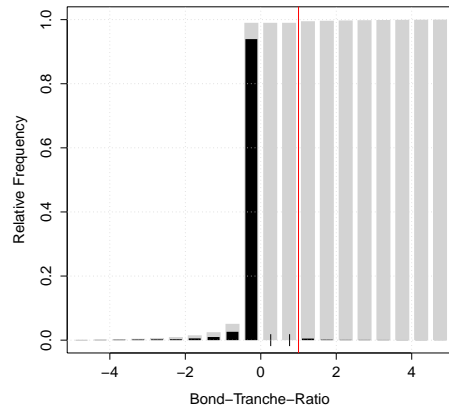
(c)  $n_B = 10$



(d)  $n_B = 25$



(e)  $n_B = 50$



(f)  $n_B = 100$

**Figure 2:** Relative (black) and cumulative (gray) frequency of bond-tranche-ratios for different levels of the number of bond factors  $n_B$ .

$\rho_B$	0.01	0.02	...	...	...	0.20
$\rho_T$	0.60	0.61	...	...	...	0.95
$\omega$	0.05	0.10	...	...	...	0.95
$\lambda$	0.001	0.01	0.05	0.10	...	0.30
$N$	1	10	100	500	1000	5000
$n_B$	1	5	10	25	50	100

**Table 8:** Simulation design levels. A combination of one value from each row represents one simulation configuration.

tween  $-1$  and zero which implies that the undiversifiable level of tranche portfolio variance<sup>7</sup> is much higher than in the bond portfolio. By contrast, as the number of bond portfolio factors rises, the possibility to replicate it by means of a tranche portfolio becomes lower and lower very fast. For instance, for  $n_B = 5$ , which is certainly a lower bound even for bond portfolios, almost 90% of all examined constellations imply consistently higher (and thus undiversifiable) tranche portfolio variances.

## 5 Conclusion

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<sup>7</sup>From (??) it is obvious that this is  $\lim_{n_T \rightarrow \infty} \mathbb{V}[L_T] = \text{LGD}^2 (\Phi_2(\cdot; \rho_T \omega) - \lambda^2)$ .

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# Appendix

## A Loan Equivalent Representation

Conditional expected loss as a function of  $M$  contains all systematic information on a defaultable security. Calculating the loss distribution of a CDO tranche usually requires to simulate the whole underlying asset pool. To reduce the computational burden CDOs are frequently treated as bonds. However, as showed by Donhauser et al. (2010), special adjustments are necessary for such a “bond representation” to work properly. In addition to computational simplification a bond representation simplifies comparisons with corporate or government bonds. Subsequently, we outline one possible procedure.

We want to replicate an original expected loss profile on a grid  $\mathbf{M}$

$$(M, \mathbb{E}[L^0 | M])_{M \in \mathbf{M}}$$

where

$$\mathbb{E}[L^0 | M] = \lambda^0(M) \cdot \text{LGD}^0(M)$$

i.e., the conditional PDs and LGDs may be arbitrary functions of  $M$ .

We replicate it by means of a simple bond model with conditional PDs and fixed LGD:

$$\mathbb{E}[L | M] = \lambda(M) \cdot \text{LGD}$$

Note that this implies that  $\text{LGD} = 1$  whenever  $\mathbb{E}[L^0 | M]$  ranges from zero to unity.

Taking expectations and solving for the PD we obtain

$$\lambda = \frac{\mathbb{E}[L^0]}{\text{LGD}}$$

Thus, our approximation equation is

$$\begin{aligned} \mathbb{E}[L^0 | M] &= \mathbb{E}[L | M] + \varepsilon \\ &= \lambda(M) \cdot \text{LGD} + (1 - \lambda(M)) \cdot 0 + \varepsilon \\ &= \lambda(M) \cdot \text{LGD} + \varepsilon \end{aligned} \tag{21}$$

for any  $M$  and  $\varepsilon$  is an approximation error.

Based on these assumptions we calibrate a Gaussian single risk factor model now. First, the asset value process at maturity

$$R = \sqrt{\rho}M + \sqrt{1 - \rho}U \tag{22}$$



Second, default occurs when

$$R < c \tag{23}$$

where  $c$  is a constant called “default threshold”.

Third, the probability of default

$$\mathbb{P}[R < c] = \lambda = \frac{\mathbb{E}[L^0]}{\text{LGD}} \tag{24}$$

Fourth, the conditional probability of default

$$\mathbb{P}[R < c | M] = \lambda(M) = \Phi\left(\frac{\Phi^{-1}(\lambda) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \tag{25}$$

Thus, the original profile is explained by

$$\mathbb{E}[L^0 | M] = \Phi\left(\frac{\Phi^{-1}(\lambda) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right) \cdot \text{LGD} + \varepsilon \tag{26}$$

The only remaining parameter is  $\rho$  which has to be chosen so that the approximation error  $\varepsilon$  is small.

Finally, note that a crucial restriction of our approximation is fulfilled per construction. The original as well as the approximating profile have equal expected loss:

$$\mathbb{E}[\mathbb{E}[L^0 | M]] = \mathbb{E}[L^0] \tag{27}$$

and

$$\begin{aligned} \mathbb{E}[\mathbb{E}[L | M]] &= \text{LGD} \cdot \mathbb{E}[\lambda(M)] \\ &= \text{LGD} \cdot \lambda \\ &= \text{LGD} \cdot \frac{\mathbb{E}[L^0]}{\text{LGD}} \\ &= \mathbb{E}[L^0] \end{aligned} \tag{28}$$

## B Relative Variance

This section is to relate the relative variance  $\tilde{\sigma}$  to asset correlation  $\rho$  which is a standard measure of systematic risk sensitivity. We derive a closed form expression for  $\tilde{\sigma}$  for single names in the Gaussian single risk factor model with fixed LGD.

Above we defined relative variance as follows:

$$\tilde{\sigma} = \frac{\mathbb{V}[\mathbb{E}[L | M]]}{\mathbb{V}[\mathbb{E}[L | M]_{\max}]} \tag{29}$$

The denominator of this ratio is already known from the main text. It is given by

$$\mathbb{V}[\mathbb{E}[L | M]_{\max}] = \text{LGD} \cdot \mathbb{E}[L] - \mathbb{E}[L]^2 \quad (30)$$

The numerator can be written as follows

$$\begin{aligned} \mathbb{V}[\mathbb{E}[L | M]] &= (\text{LGD})^2 \cdot \mathbb{V}[\lambda(M)] \\ &= (\text{LGD})^2 \cdot (\mathbb{E}[\lambda(M)^2] - \lambda^2) \\ &= (\text{LGD})^2 \cdot \mathbb{E}[\lambda(M)^2] - (\text{LGD})^2 \cdot \left(\frac{\mathbb{E}[L]}{\text{LGD}}\right)^2 \\ &= (\text{LGD})^2 \cdot \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \mathbb{E}[L]^2 \end{aligned} \quad (31)$$

The bivariate normal distribution function  $\Phi_2(\cdot)$  comes from the observation that

$$\begin{aligned} \mathbb{E}[\lambda(M)^2] &= \mathbb{E}[\mathbb{P}[R_i < \Phi^{-1}(\lambda), R_j < \Phi^{-1}(\lambda) | M]] \\ &= \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) \end{aligned} \quad (32)$$

Putting all parts together we obtain

$$\begin{aligned} \tilde{\sigma} &= \frac{(\text{LGD})^2 \cdot \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \mathbb{E}[L]^2}{\text{LGD} \cdot \mathbb{E}[L] - \mathbb{E}[L]^2} \\ &= \frac{\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \lambda^2}{\lambda - \lambda^2} \end{aligned} \quad (33)$$

which has an intuitive interpretation. It is the difference between the joint PD and the joint PD with independence divided by the difference between the maximum joint PD and the joint PD with independence. The joint PD with independence,  $\lambda^2$ , is also the minimum joint PD as long as  $\rho \geq 0$  is required.

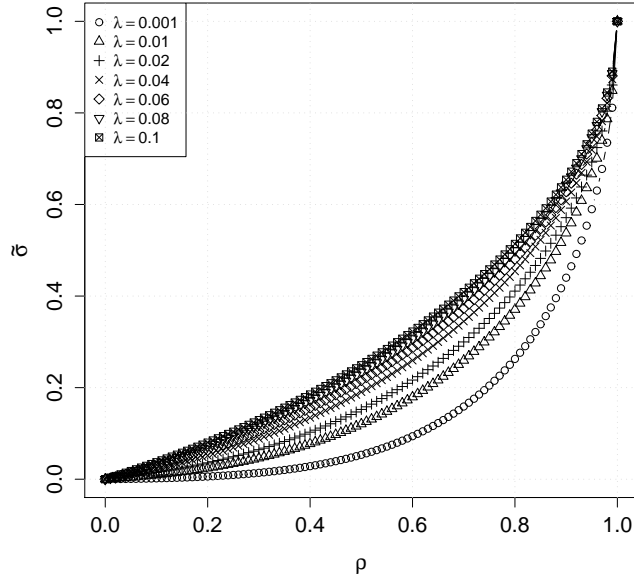
Now note that

$$\lim_{\rho \rightarrow 1} \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) = \lambda = \frac{\mathbb{E}[L]}{\text{LGD}} \quad (34)$$

and thus  $\lim_{\rho \rightarrow 1} (\tilde{\sigma}) = 1$ .

Finally,

$$\lim_{\rho \rightarrow 0} \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) = \lambda^2 = \left(\frac{\mathbb{E}[L]}{\text{LGD}}\right)^2 \quad (35)$$



**Figure 3:** Relative variance  $\tilde{\sigma}$  as a function of  $\rho$ .

and thus  $\lim_{\rho \rightarrow 0} (\tilde{\sigma}) = 0$ .

This shows that  $\rho$  and  $\tilde{\sigma}$  are comonotonic and have the same limits.

A graphical illustration of this result for different levels of  $\lambda$  is shown in Figure 3. We clearly see that the lower  $\lambda$  the greater the distance between  $\rho$  and  $\tilde{\sigma}$ . For example, for  $\lambda = 0.001$  a level of  $\rho = 0.8$  corresponds to  $\tilde{\sigma} = 0.27$ .

## C Portfolio Variance

Substituting (14) and (15) into (13) we obtain

$$\begin{aligned}
\mathbb{V}[L] &= \mathbb{V}\left[\sum_j \frac{1}{n} L_j\right] \\
&= \sum_j \frac{1}{n^2} \mathbb{V}[L_j] + \sum_{j < j'} 2 \cdot \frac{1}{n^2} \text{Cov}(L_j, L_{j'}) \\
&= \frac{1}{n^2} \sum_j \text{LGD}^2 [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \lambda^2] \\
&\quad + \text{LGD}^2 \left[ \frac{1}{N} (\lambda - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho)) \right] \\
&\quad + 2 \cdot (n-1) \cdot \frac{1}{2} (n-1) \cdot \frac{1}{n^2} \text{Cov}(L_j, L_{j'}) \\
&= \text{LGD}^2 \cdot \frac{n}{n^2} [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \lambda^2] \\
&\quad + \text{LGD}^2 \cdot \frac{n}{n^2} \left[ \frac{1}{N} (\lambda - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho)) \right] \\
&\quad + \text{LGD}^2 \cdot (n-1) \cdot \frac{n}{n^2} \cdot [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho\omega) - \lambda^2] \\
&= \text{LGD}^2 \cdot \frac{1}{n} [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \lambda^2] \\
&\quad + \text{LGD}^2 \cdot \frac{1}{n} \left[ \frac{1}{N} (\lambda - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho)) \right] \\
&\quad + \text{LGD}^2 \cdot \frac{1}{n} \cdot (n-1) \cdot [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho\omega) - \lambda^2]
\end{aligned} \tag{36}$$

Finally, we obtain

$$\begin{aligned}
\mathbb{V}[L] &= \text{LGD}^2 \cdot [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho\omega) - \lambda^2] \\
&\quad + \text{LGD}^2 \cdot \frac{1}{n} \left[ \frac{1}{N} (\lambda - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho)) \right] \\
&\quad + \text{LGD}^2 \cdot \frac{1}{n} \cdot [\Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho) - \Phi_2(\Phi^{-1}(\lambda), \Phi^{-1}(\lambda); \rho\omega)]
\end{aligned} \tag{37}$$