

CDS pricing with long memory via fractional Lévy processes

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Abstract

In this paper, we consider spread rates of credit default swaps (CDSs) in a long memory fractional Lévy setting, i.e. where interest and hazard rates are driven by processes whose autocovariance functions decrease very slowly over time. Empirically, this property can be found in many variables like interest and hazard rates, but the usually applied Markovian models are unable to reflect this. Using earlier results on conditional distributions of fractional Lévy processes, we carry out an extensive analysis of parameter sensitivities useful for researchers and practitioners alike and derive an analytical pricing formula for CDS contracts. A first empirical application is provided as well.

Keywords: Long memory; credit default swap; fractional Lévy process; long range dependence; fractional Brownian motion; CDS pricing.

1. Introduction

One of the most simplified frameworks for fixed income markets has been developed and proposed by Vasicek (1977). Basically this model assumes the existence of an instantaneous short rate $r = (r(t))_{t \geq 0}$, i.e. heuristically let Δt be the

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length of an infinitesimal small time interval, then the respective risk-free interest rate is given by $r(t)\Delta t$. In particular, the model is driven by a Brownian motion $B = (B(t))_{t \geq 0}$ which lies in the class of semimartingales. The short rate process is described by a stochastic differential equation (sde) of the Gaussian Ornstein–Uhlenbeck type:

$$dr(t) = (k - ar(t))dt + \sigma dB(t), \quad t \geq 0, \quad r(0) \in \mathbb{R}^+,$$

for $k, a, \sigma > 0$, which results in a stationary and mean-reverting process with long-term mean given by k/a . To simplify things even further one can assume that the dynamics of r are already given under a risk-neutral pricing measure \mathcal{Q} and therefore it follows that prices of (non-defaultable) zero coupon bonds are given by

$$B(t, T) = E^{\mathcal{Q}}[e^{-\int_t^T r(v)dv} | r(s), 0 \leq s \leq t], \quad 0 \leq t \leq T.$$

Due to the fact that B and therefore r are Markov processes, the above pricing formula can be evaluated fast and efficient. Of course, as a Gaussian process, r can also take negatives values which is a major disadvantage of the Vasicek model. However one can always shift and scale the model to let the probability of a negative r become arbitrarily small. Analogously, one can introduce a hazard rate process λ to allow for a possible default of the above zero coupon bonds which leads to a credit market setting, cf. for example [Frey and Backhaus \(2008\)](#) or [Schönbucher \(2003\)](#).

Over time, many potential extensions and related models have been proposed, e.g. [Cox et al. \(1985\)](#) (square root processes to obtain positive short rates), [Hull and White \(1990\)](#) (time dependent coefficient functions for the Vasicek sde), [Eberlein and Raible \(1999\)](#) (general Lévy models to obtain distributions different from the Gaussian one) and [Duffie et al. \(2000, 2003\)](#) (affine Markov processes). In general, these processes offer many analytical features and provide a convenient way to include market and idiosyncratic information for the purpose of credit risk modeling (cf. [Hamerle et al., 2012](#)).

All these approaches however, are not able to capture some empirical properties of interest rates. Statistical observations (cf. [Henry and Zaffaroni, 2003](#); [Backus and Zin, 1993](#)) suggest that the Markov structure inherent in the models above is not able to reflect the situation at the real markets as there is evidence of so-called ‘long range dependence’ (or ‘long memory’) which basically means that the autocovariance function of the respective time series does decline very slowly (cf. Sec. 4 of [Backus and Zin \(1993\)](#) for the short rate in above model).

Furthermore, in the literature, several concepts specify the hazard rate as a function comprising macroeconomic as well as firm-specific variables. For example, [Duffie et al. \(2009\)](#) chose the 3-month Treasury bill rate and the trailing 1-year return on the S&P 500 index to integrate macroeconomic information. Empirical evidence on the long memory property of interest rates has already been

cited. Moreover, equity processes are also partially known to exhibit increments that are positively correlated over the long run (cf. Henry, 2002). In general, there are many macroeconomic and financial variables which are assumed to have long range dependence (cf. Baillie, 1996). These considerations give rise to the assumption that beside short also hazard rates incorporate long memory and again, classical Markovian models are not able to capture this fact.

To overcome this particular structural drawback, fractional models have been suggested (cf. Ohashi, 2009; Fink *et al.*, 2012; Biagini *et al.*, 2013; Fink, 2013). However the past work concentrated on discussions of suitable no-arbitrage settings and conditional distributions of the appearing fractional processes. Although Biagini *et al.* (2013) derive explicit formulas for derivative pricing, credit default swaps (CDSs) have not yet been fully analyzed in particular. Hao *et al.* (2014) is a first step, however they only focus on fractional Brownian motion (fBm) and a two-firm contagion model. Our approach, which is based on the much more general setting of Biagini *et al.* (2013) and especially Fink (2013), covers also the fractional Lévy case.

The central aim of this paper is to develop an analytical formula for the valuation of CDS contracts by working in the mentioned fractional setting and therefore explicitly incorporating long memory. We shall furthermore provide an explicit sensitivity analysis for standard defaultable bonds and CDS contracts alike.

The remainder of this paper is organized as follows: A brief introduction to the notion of long memory and the use of Molchan–Golosov kernels generating fractional Lévy processes is presented in Sec. 2. In Sec. 3, which summarises earlier results of Fink (2013), a fractional market model is described and pricing formulas for (defaultable) zero coupon bonds are stated. While Sec. 4 provides an extensive analysis of the various model parameters in the fractional setting, Sec. 5 proposes new results on the spread rates of CDSs in our general fractional credit setting. Furthermore, the model-implied CDS term structure is fitted to real market data.

1.1. Notation

For the whole paper we shall assume a given complete probability space $(\Omega, \mathcal{F}, \mathcal{Q})$. Denote by $L^2(\Omega)$ the space of square integrable random variables, i.e. random variables with finite variance on Ω . For a family of random variables $(X(i))_{i \in I}$, I some index set, let $\overline{\sigma\{X(i), i \in I\}}$ denote the completion (with respect to sets of measure zero) of the generated σ -algebra.

For $A \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, the spaces of integrable and square integrable functions $f : A \rightarrow \mathbb{R}^{n \times n}$ are denoted by $L^1(A, \mathbb{R}^{n \times n})$ and $L^2(A, \mathbb{R}^{n \times n})$. If the dimension of the image space is $n = 1$ we shall just write $L^1(A)$ and $L^2(A)$. Moreover for $T > 0$, $\|\cdot\|$ is the L^2 -norm and $\langle \cdot, \cdot \rangle$ the corresponding Euclidian scalar product on $L^2([0, T])$.

2. Molchan–Golosov Fractional Lévy Processes

Before we start with fractional Lévy processes, the core objects of this paper, we need to fix the necessary notions and definitions. Therefore we shall give a brief overview of fractional analysis.

2.1. Fractional calculus

Fractional integrals and derivatives can be used to generate fractional processes with dependent increments by convolution of a stochastic process which has independent returns. However there are various ways to do this and not all approaches lead to the same result. For example fractional Lévy processes by so called Mandelbrot–Van Ness kernels (cf. Mandelbrot and Van Ness, 1968) have been introduced by Marquardt (2006) while Tikanmäki and Mishura (2011) and Fink (2013) considered Molchan–Golosov kernels (cf. Molchan and Golosov, 1969; Kleptsyna *et al.*, 1999; Norros *et al.*, 1999; Decreusefond and Üstünel, 1999). In general, both approaches lead to different processes and have their own advantages and shortcomings. The Mandelbrot–Van Ness definition leads to stationary processes while the Molchan–Golosov kernels allow for subordinators, i.e. a.s. increasing processes. In this paper, we shall focus on the later ones out of a simple reasoning: in most cases, interest rates and especially hazard rates should be positive (excluding cases like Japan and the recent negative deposit rate in the Eurozone).

A very detailed survey on fractional calculus can be found in Samko *et al.* (1993). The main concept is also closely related to Riemann–Stieltjes integration and stochastic calculus and we refer the interested reader to Zähle (1998, 2001). From now on we shall work on the compact interval $[0, T]$ for some fixed $T > 0$.

Definition 2.1. For a constant $0 < d < 1$ and $f \in L^1([0, T])$ define the *fractional Riemann–Liouville integral of f of order d* with finite time horizon by the expression

$$(I_T^d f)(s) = \frac{1}{\Gamma(d)} \int_s^T f(r)(r-s)^{d-1} dr, \quad 0 < s < T, \quad (2.1)$$

where Γ shall be the Gamma-function.

For $f \in L^1([0, T])$ the fractional integrals always exists almost everywhere, cf. (7) of Zähle (1998). The fractional derivative with finite time horizon however is another story. For $\alpha \in (0, 1)$ it can be introduced as an inverse operation to fractional integration, but its existence is much more sophisticated. However we will not worry about this question in the present paper and consider only situations where the expression exists and is well-defined.

Definition 2.2. For a constant $0 < d < 1$ let $g \in L^1([0, T])$ such that there exists $\psi_g \in L^1([0, T])$ satisfying

$$g(s) = I_{T-}^d(\psi_g(\cdot))(s), \quad 0 < s < T. \tag{2.2}$$

Then define the *fractional Riemann–Liouville derivative of g of order d* by

$$(D_{T-}^d g)(u) = \frac{1}{\Gamma(1-d)} \left(\frac{g(u)}{(T-u)^d} + d \int_u^T \frac{g(u) - g(s)}{(s-u)^{d+1}} ds \right), \quad 0 < u < T. \tag{2.3}$$

As it is convention, we shall often write $I_{T-}^{-d} = D_{T-}^d$. For $d = 0$ we set $I_{T-}^d = D_{T-}^d = id$.

2.2. Convolution

Motivated by interest and hazard rates which are usually positive processes we will from now on focus on the class of so-called Molchan–Golosov fractional Lévy processes which are introduced by a Molchan–Golosov integration kernel (cf. Molchan and Golosov, 1969) and have been considered in Tikanmäki and Mishura (2011) and Fink (2013). This class includes a certain type of fractional subordinators, i.e. a.s. increasing processes, which can be used to model positive processes (e.g. Bender and Marquardt, 2009 who considered a Black–Scholes model with fractional volatility). Furthermore we want to mention that we will restrict our considerations in this paper to a fractional parameter between zero and 0.5 which reflects exactly the long range dependence case. In the basic work of Tikanmäki and Mishura (2011) and Fink (2013) a more general definition is presented. We shall consider a given multivariate square-integrable Lévy process $\mathbf{L} = (\mathbf{L}(t))_{t \in [0, T]} = (L^1(t), \dots, L^n(t))_{t \in [0, T]}^\top$, for $n \in \mathbb{N}$ and $T > 0$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathcal{Q})$ satisfying the usual conditions (right-continuity and completeness). For the rest of this section we will state the necessary results from Fink (2013).

For $f \in L^2([0, T], \mathbb{R}^{n \times n})$ and $d = (d(1), \dots, d(n))^\top \in [0, \frac{1}{2}]^n$ define for $s \in [0, T]$ the convolution operator

$$z^d(f, s) := \begin{pmatrix} c_{d(1)} s^{-d(1)} I_{T-}^{d(1)}((\cdot)^{d(1)} f_{11}(\cdot))(s) & \dots & c_{d(n)} s^{-d(n)} I_{T-}^{d(n)}((\cdot)^{d(n)} f_{1n}(\cdot))(s) \\ \vdots & \ddots & \vdots \\ c_{d(1)} s^{-d(1)} I_{T-}^{d(1)}((\cdot)^{d(1)} f_{n1}(\cdot))(s) & \dots & c_{d(n)} s^{-d(n)} I_{T-}^{d(n)}((\cdot)^{d(n)} f_{nn}(\cdot))(s) \end{pmatrix}$$

using fractional integration as defined in (2.1), where for $1 \leq j \leq n$ the one-dimensional function f_{ij} is the ij th component of f and

$$c_{d(j)} = \left(\frac{(2d(j) + 1)\Gamma(d(j) + 1)\Gamma(1 - d(j))}{\Gamma(1 - 2d(j))} \right)^{\frac{1}{2}}.$$

One can furthermore show that $z^d(f, \cdot) \in L^2([0, T], \mathbb{R}^{n \times n})$.

Definition 2.3. [Definition 3.1 of Fink (2013)] For $d = (d(1), \dots, d(n))^\top \in [0, \frac{1}{2}]^n$ and $n \in \mathbb{N}$ we define the kernel $z^d(\hat{\mathbf{1}}_{[0,t]}, \cdot) : [0, T] \rightarrow \mathbb{R}^{n \times n}$ using

$$\hat{\mathbf{1}}_{[0,t]}(\cdot) := \begin{pmatrix} \mathbf{1}_{[0,t]}(\cdot) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{1}_{[0,t]}(\cdot) \end{pmatrix}.$$

Then a Molchan–Golosov fractional Lévy process (MG-fLp)

$$\mathbf{L}^d = (\mathbf{L}^d(t))_{t \in [0, T]} = (L^{d(1)}(t), \dots, L^{d(n)}(t))_{t \in [0, T]}^\top$$

is defined in the $L^2(\Omega)$ -sense via

$$\mathbf{L}^d(t) = \int_0^t z^d(\hat{\mathbf{1}}_{[0,t]}, s) d\mathbf{L}(s), \quad t \in [0, T]. \tag{2.4}$$

Define for $d \in [0, \frac{1}{2}]^n$ and suitable $f : [0, T] \rightarrow \mathbb{R}^{n \times n}$ (such that the expression below exists) the deconvolution operator for $s \in [0, T]$ by

$$z^d(f, s) := \begin{pmatrix} c_{-d(1)}^{-1} s^{d(1)} I_{T-}^{d(1)}((\cdot)^{-d(1)} f_{11}(\cdot)(\cdot))(s) & \dots & c_{-d(n)}^{-1} s^{d(n)} I_{T-}^{d(n)}((\cdot)^{-d(n)} f_{1n}(\cdot)(\cdot))(s) \\ \vdots & \ddots & \vdots \\ c_{-d(1)}^{-1} s^{d(1)} I_{T-}^{d(1)}((\cdot)^{-d(1)} f_{n1}(\cdot)(\cdot))(s) & \dots & c_{-d(n)}^{-1} s^{d(n)} I_{T-}^{d(n)}((\cdot)^{-d(n)} f_{nn}(\cdot)(\cdot))(s) \end{pmatrix}.$$

Example 2.4. [Example 3.1 of Fink (2013)] Let $n = 1$ in Definition 2.3.

- (i) For $d = 0$ we get by definition $\mathbf{L}^d = \mathbf{L}$.
- (ii) Choosing as driving Lévy process a standard Brownian motion, we get a classical fBm on $[0, T]$ like in Samorodnitsky and Taqqu (1994).
- (iii) For a strictly increasing subordinator as driving Lévy process we obtain a fractional subordinator in the sense of Example 1 of Bender and Marquardt (2009), which is itself a.s. increasing.

Proposition 2.5. [Proposition 3.1 of Fink (2013)] For $s, t \in [0, T]$ we have for the mean-value and autocovariance function

$$(i) E[\mathbf{L}^d(t)] = \int_0^t z^d(\hat{\mathbf{1}}_{[0,t]}, s) ds \cdot E[\mathbf{L}(1)].$$

(ii) $\text{Cov}[\mathbf{L}^d(t), \mathbf{L}^d(s)] = \frac{1}{2}(c_{d(i),d(j)} \text{Cov}[L^i(1), L^j(1)](t^{d(i)+d(j)+1} + s^{d(i)+d(j)+1} - |t - s|^{d(i)+d(j)+1}))_{1 \leq i, j \leq n}$, where

$$c_{d(i),d(j)} = \frac{\sqrt{\Gamma(2d(i) + 2) \sin(\pi(d(i) + \frac{1}{2}))} \sqrt{\Gamma(2d(j) + 2) \sin(\pi(d(j) + \frac{1}{2}))}}{\Gamma(d(i) + d(j) + 2) \sin(\pi(d(i) + d(j) + 1)/2)}.$$

In the literature there are various definitions for ‘long range dependence’ and a good overview can be found in Samorodnitsky (2007). As discussed there, early considerations of Mandelbrot, e.g. Mandelbrot (1965) and Mandelbrot and Wallis (1968) were motivated by studies on the flow of water in the Nile river carried out by Hurst (1951, 1956). In the context of this paper, we shall specify long range dependence by the rate of decrease of the autocovariance function.

Definition 2.6. Let $X = (X(t))_{t \in \mathbb{R}}$ be a weakly stationary process and let

$$\gamma_X(h) := \text{Cov}(X(t + h), X(t)), \quad h \in \mathbb{R},$$

be its autocovariance function. The process X exhibits *long range dependence* if $d \in (0, \frac{1}{2})$ and $c_\gamma > 0$ exists such that

$$\lim_{h \rightarrow \infty} \frac{\gamma_X(h)}{h^{2d-1}} = c_\gamma. \tag{2.5}$$

Consider for $d \in [0, \frac{1}{2})$ the covariance between two increments of an one-dimensional MG-fLp L^d . Then we get by Proposition 2.5

$$\begin{aligned} \gamma_{L^d}(h) &:= \text{Cov}(L^d(k) - L^d(k - 1), L^d(k + h) - L^d(k + h - 1)) \\ &= \frac{E[L^2(1)]}{2} [|h + 1|^{2d+1} - |h - 1|^{2d+1} - 2|h|^{2d+1}], \end{aligned} \tag{2.6}$$

for $h, k \in \mathbb{N}$ with $h + k \leq T$.

Of course, we defined MG-fLps just on a compact time-set and therefore the definition above cannot be applied directly. However the finite time horizon $T > 0$ can be arbitrarily large and we have

$$[|h + 1|^{2d+1} - |h - 1|^{2d+1} - 2|h|^{2d+1}] \sim Ch^{2d-1},$$

as $h \rightarrow \infty$ for some fixed $C > 0$.

2.3. Integration

To explain our credit market setting in the next session, integration needs to be specified. For this, an $L^2(\Omega)$ -approach will be chosen (cf. Phipras and Taqqu, 2000, 2001; Marquardt, 2006; Tikanmäki and Mishura, 2011; Fink, 2013). Again, we

shall follow [Fink \(2013\)](#): Considering step functions

$$f(\cdot) = \sum_{k=1}^m a_k \mathbf{1}_{[t_k, t_{k+1})}(\cdot),$$

where $m \in \mathbb{N}$, $m \geq 1$, $0 \leq t_1 \leq \dots \leq t_m \leq T$ and $a_k \in \mathbb{R}^{n \times n}$ for $1 \leq k \leq m$, we set

$$\int_0^T f(s) d\mathbf{L}^d(s) := \sum_{k=1}^m a_k (\mathbf{L}^d(t_{k+1}) - \mathbf{L}^d(t_k)).$$

It is now easy to see, that

$$\int_0^T f(s) d\mathbf{L}^d(s) = \int_0^T z^d(f, s) d\mathbf{L}(s).$$

One can show that this property holds true for all integrands in $L^2([0, T], \mathbb{R}^{n \times n})$ as stated by the following theorem.

Theorem 2.7. [Theorem 3.1 of [Fink \(2013\)](#)] For $d = [0, \dots, d(n)]^\top \in (-\frac{1}{2}, \frac{1}{2})^n$, $n \in \mathbb{N}$, let $f \in L^2([0, T], \mathbb{R}^{n \times n})$. Then the integral $\int_0^T f(s) d\mathbf{L}^d(s)$ exists as a (componentwise) $L^2(\Omega)$ -limit of approximating step functions in Λ_T^d (also componentwise). Furthermore, we have the identity

$$\int_0^T f(s) d\mathbf{L}^d(s) = \int_0^T z^d(f, s) d\mathbf{L}(s),$$

which holds (componentwise) in $L^2(\Omega)$.

3. Fractional Credit Model

The specification of our market model generalizes the concept of the well-known *two-factor Gaussian* or *Vasicek model*. As in the fractional setting of [Biagini et al. \(2013\)](#), we adopt the structure of this reduced form approach and initially specify a finite time horizon $T^* > 0$, a complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathbb{Q})$ and a bivariate process $(r, H) := (r(t), H(t))_{0 \leq t \leq T^*}$ where H represents the *default indicator process*, i.e.

$$H(t) = \mathbf{1}_{\{\tau \leq t\}}, \quad 0 \leq t \leq T^*$$

The random variable τ denotes a $(\mathcal{F}_t)_{0 \leq t \leq T^*}$ -stopping time, the so-called *default time*. Furthermore, let $(\mathcal{H}_t)_{0 \leq t \leq T^*}$ be the filtration generated by H .

Remark 3.1. Markets driven by MG-fLps might allow arbitrage as these fractional processes are, in general, no longer semimartingales (cf. [Delbaen and](#)

Schachermayer, 1989, 1994; Bender *et al.*, 2007; Cheridito, 2003). First however, in the case of fBm, one can derive the Vasicek model (which we will consider below) from the fractional Heath–Jarrow–Morton approach of Ohashi (2009) which is based on previous work of Guasoni *et al.* (2008, 2010). Here the implementation of suitable transaction costs excluded arbitrage. The existence of an average risk-neutral measure can be proven and we can formally calculate prices of defaultable bonds or more general contingent claims under this measure as suggested in Sottinen and Valkeila (2001). Second, if we use a MG-fLp with finite variation like a fractional Poisson process, we are again in the semimartingale case. The situation of non-fBm MG-fLps with infinite variation has not yet been analyzed, however we shall include it nevertheless in our following analysis, as one can always directly model under a risk neutral measure.

Assumption 3.2. [Credit market structure; cf. Frey and Backhaus (2008), Assumption 3.1]

- (i) There is a subfiltration $(\mathcal{G}_t)_{0 \leq t \leq T^*}$ of $(\mathcal{F}_t)_{0 \leq t \leq T^*}$, with

$$\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t, \quad 0 \leq t \leq T^*$$

λ is a $(\mathcal{G}_t)_{0 \leq t \leq T^*}$ -progressive process, which describes the intensity of H (cf. Corollary 5.1.5 of Bielecki and Rutkowski (2002)), satisfying $\int_0^t \lambda(s) ds < \infty$ a.s. for all $0 \leq t \leq T^*$ and

$$\mathcal{Q}(\tau > t | \mathcal{G}_t) = E[1 - H(t) | \mathcal{G}_t] = \exp\left\{-\int_0^t \lambda(s) ds\right\}. \quad (3.1)$$

For all bounded \mathcal{G}_∞ -measurable random variables η , $\mathcal{G}_\infty := \bigvee_{0 \leq t \leq T^*} \mathcal{G}_t$, we have

$$E[\eta | \mathcal{F}_t] = E[\eta | \mathcal{G}_t]. \quad (3.2)$$

- (ii) Under the risk-neutral pricing measure \mathcal{Q} , the price of an arbitrary \mathcal{F}_T -measurable claim $X \in L^1(\Omega)$ with maturity $0 \leq T \leq T^*$ at time $0 \leq t \leq T$ is given by $V(t, T) := E[X | \mathcal{F}_t]$.

These standard assumptions seem rather technical at first glance. Yet, the filtration $(\mathcal{F}_t)_{0 \leq t \leq T^*}$ basically reflects the market information at time t , i.e. the investor knows about the state variables r and λ as well as the default indicator process H at time t . Furthermore, for $0 \leq t + \Delta t \leq T^*$, we have

$$\mathcal{Q}(t < \tau \leq t + \Delta t | \mathcal{F}_t) \approx \mathbf{1}_{\{\tau > t\}} \lambda(t) \Delta t$$

which means that $\lambda(t)\Delta t$ is approximately the conditional probability under \mathcal{Q} of defaulting in a small time interval after t , given the survival up to time t . However, recall that \mathcal{Q} is a risk-neutral measure, and this probability does not directly reflect the “real” default probability.

Using Assumption 3.2(ii) and invoking e.g., Proposition 3.1 of Lando (1998), the price of a defaultable zero coupon bond $\bar{B}(t, T)$ for $0 \leq t \leq T \leq T^*$ can be calculated as follows:

$$\bar{B}(t, T) = E[e^{-\int_t^T r(s)ds} \mathbf{1}_{\{\tau > T\}} | \mathcal{F}_t] = \mathbf{1}_{\{\tau > t\}} E[e^{-\int_t^T [r(s) + \lambda(s)] ds} | \mathcal{G}_t]. \quad (3.3)$$

This result is very useful because it implies that for pricing purposes we only need to focus on the bivariate process (r, λ) and can ignore the default indicator process.

To account for long range dependence in interest and hazard rates, we now propose a two-dimensional fractional Vasicek model (cf. Vasicek, 1977) for (r, λ) , based on the MG-fLps introduced above.

Assumption 3.3. [Fractional Vasicek credit market; Version of Fink (2013), Assumption 5.2] For $d = (d(1), \dots, d(n))^T \in [0, \frac{1}{2}]^n$, $n \in \mathbb{N}$, take $k : [0, T^*] \rightarrow \mathbb{R}^n$ and $a : [0, T^*] \rightarrow \mathbb{R}^{n \times n}$ locally integrable, σ non-singular for every $t \in [0, T^*]$, with σ_{ij} and $(\sigma)_{ij}^{-1}$ of bounded $p(j)$ -variation for some $0 < p(j) < 1/[1 - d(j)]$ for all $1 \leq i, j \leq n$, and fixed weights $\theta, \phi \in (\mathbb{R}_+)^n$. Let the process $\mathcal{L}^d := (\mathcal{L}^d(t))_{t \in [0, T^*]}$ be the unique solution of the Vasicek sde

$$d\mathcal{L}^d(t) = [k(t) - a(t)\mathcal{L}^d(t)]dt + \sigma(t)d\mathbf{L}^d(t), \quad t \in [0, T^*], \quad \mathcal{L}^d(0) \in \mathbb{R}^n$$

which exists according to Proposition 4.9 of Fink (2013). Then set

$$r(t) = \langle \theta, \mathcal{L}^d(t) \rangle \quad \text{and} \quad \lambda(t) = \langle \phi, \mathcal{L}^d(t) \rangle, \quad t \in [0, T^*]. \quad (3.4)$$

Again, the model assumptions appear to be rather technical. However, if we take, for example, a fractional subordinator as the driving power behind the Vasicek sde, it suffices to choose continuous functions a, k and σ in order to satisfy the conditions of Assumption 3.3. Moreover, in the general setup above, we have $\mathcal{G}_t = \sigma \{ \mathbf{L}^d(s), s \in [0, t] \}$ for all $t \in [0, T^*]$, if $\theta + \phi$ is componentwise not zero.

Using pricing formula (3.3), we can explicitly calculate the value of a defaultable zero coupon bond in the *fractional Vasicek credit market*.

Theorem 3.4. [Theorem 5.3 of (Fink, 2013)] Let $0 \leq t \leq T \leq T^*$ and define $D(t, T) := \int_t^T e^{-\int_t^s a(v)dv} ds$. If

$$E \left[\exp \left\{ - \left\langle \theta + \phi, \int_t^T D(v, T) \sigma(v) d\mathbf{L}^d(v) \right\rangle \right\} \right] < \infty$$

then, setting $h(\cdot) := D(\cdot, T)\sigma(\cdot)$, the price of a defaultable zero coupon bond in the fractional Vasicek credit market can be determined as follows:

$$\begin{aligned} \bar{B}(t, T) &= \mathbf{1}_{\{\tau > t\}} E[e^{-\int_t^T [r(s) + \lambda(s)] ds} | \mathcal{G}_t] \\ &= \mathbf{1}_{\{\tau > t\}} \exp\left\{-\left\langle \theta + \phi, D(t, T)\mathcal{L}^d(t) + \int_t^T D(v, T)k(v)dv \right\rangle\right\} \\ &\quad \times \exp\left\{-\left\langle \theta + \phi, \int_0^t z_*^{-d}(\mathbf{1}_{[0, t]}z^d(h\mathbf{1}_{[t, T]}, \cdot), v)d\mathbf{L}^d(v) \right\rangle\right\} \\ &\quad \times \exp\left\{\int_t^T \psi(z^d(h\mathbf{1}_{[t, T]}, v))^\top i(\theta + \phi)dv\right\}. \end{aligned} \tag{3.5}$$

4. Parameter Sensitivity

In this section, we want to consider the parameter sensitivities in the fractional credit setting. Previously, we introduced fractional credit markets driven by fractional Vasicek dynamics. In the classical work of Vasicek (1977) the drawback of getting a negative short rate r with positive probability was justified by the significant advantage of having a market model that provides easily tractable bond prices with analytical formulas. As mentioned, one can always shift and scale the model to make the probability of a negative short rate as small as possible. For simplicity, in the following analysis we shall set $\phi = 0$ in (3.4) since obviously the formula in Theorem 3.4 is somewhat symmetric with respect to interest and hazard rates.

Firstly, we chose a one-dimensional fractional Brownian motion as driving power in Assumption 3.3, which this leads to a fractional (Brownian) Vasicek model (cf. Fink et al., 2012) and includes the classical Vasicek model by setting $d = 0$. However in comparison to Vasicek (1977) the numerics have become more difficult. Still, it is the natural extension of the classical model and allows long range dependence in the increments of the short rate.

Secondly, we take a fractional Poisson model in Assumption 3.3 with intensity $\eta > 0$ (cf. Example 5.6 of Fink, 2013). This setting addresses the above mentioned drawback of the Gaussian setting: When using fractional subordinators as driving processes it is ensured that the short rate r cannot become negative. Also the model still allows for fairly explicit calculations of zero coupon bond prices. To sum things up, we consider (cf. Theorem 3.4):

- a fractional Brownian model with $r(0), k, \sigma \geq 0$ and $a > 0, d \in [0, \frac{1}{2})$ and

$$B(0, T) = \exp\left\{-D(0, T)r(0) - k \int_0^T D(v, T)dv\right\}$$

$$+ \frac{\sigma^2}{2} \left\| z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v)\mathbf{1}_{[0, T]}(\cdot) \right\|^2 \Big\}$$

- a fractional Poisson model with $r(0), k, \sigma \geq 0$ and $a, \eta > 0, d \in [0, \frac{1}{2})$ and

$$B(0, T) = \exp \left\{ -D(0, T)r(0) - k \int_0^T D(v, T)dv + \eta \int_0^T \exp(-\sigma z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v))dv \right\}.$$

The rest of this section will be dedicated to a detailed analysis of the bond price dynamics in both models since in practical considerations, parameter sensitivities play an important role.

For example in the classical Black–Scholes model of **Black and Scholes (1973)** these sensitivities are captured by the so-called *greeks* which are basically the derivatives with respect to the individual parameters. Market participants can apply the greeks to carry out a *ceteris paribus* analysis and approximate how the prices would change under certain assumptions. In the Black–Scholes model the greeks also play an important role when building hedging strategies. Of course these considerations are only valid if the model assumptions are. Therefore such derivatives should be used with care.

We will carry out this study by letting one parameter vary while the rest will be fixed for the time being. For good visibility, reference parameters shall be $r(0) = 0.1, a = 4, k = 1$ and $\sigma = 1$. For the fractional Poisson model the reference intensity shall be $\eta = 1$. The fractional parameter d will take the values $0, 0.1, 0.25$ and 0.45 .

Denote for the rest of this section the zero coupon bond price in the fractional Brownian model by $B^{\text{fBm}}(0, T)$ and in the fractional Poisson model by $B^{\text{Poi}}(0, T), 0 \leq T \leq T^*$. Considering the bond price as a function with respect to a certain parameter will be denoted by subscription, e.g. $B_{r(0)}^{\text{fBm}}(0, T)$.

4.1. Parameter sensitivity with respect to $r(0)$

At $T = 0$ all bond prices are equal to 1 but afterwards a higher start interest rate $r(0)$ also leads to a lower bond price as can be seen in Figs. 1 and 2.

For higher maturities T this effect becomes weaker since the weighting factor $D(0, T)$ in the pricing formula is bounded by a^{-1} . For all $d \in [0, \frac{1}{2})$ the absolute influence of $r(0)$ on the bond price is equal while the relative influence varies. The derivative with respect to $r(0)$ exists and is given by

$$\frac{\partial B_{r(0)}^{\text{fBm}}(0, T)}{\partial r(0)} = -D(0, T) \cdot B_{r(0)}^{\text{fBm}}(0, T),$$

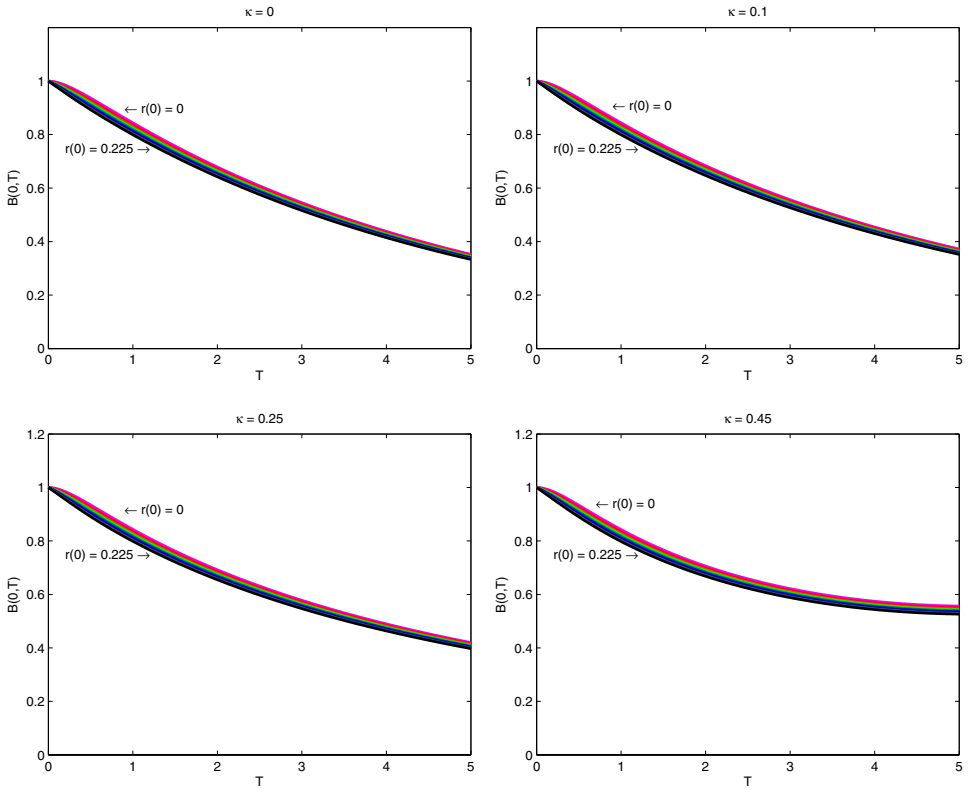


Fig. 1. Bond prices $B^{\text{Bm}}(0, T)$ in the fractional Brownian model for varying $r(0)$, maturity T and fractional parameter d , using constant coefficients $a = 4$, $k = 1$ and $\sigma = 1$. $d = 0$ corresponds to the classical Brownian Vasicek model. In particular, $r(0)$ increases by steps of size 0.025 from 0 to 0.225.

$$\frac{\partial B_{r(0)}^{\text{Poi}}(0, T)}{\partial r(0)} = -D(0, T) \cdot B_{r(0)}^{\text{Poi}}(0, T).$$

4.2. Parameter sensitivity with respect to a

As can be seen in Fig. 3, the influence of the parameter a is more difficult. The reason behind this is the fact that $D(\cdot, T)$ (and therefore a) is also involved via the fractional integration in the bond price formulas.

However it can be seen from these formulas that the impact of the parameter a is still mostly monotone. If the probability of getting a negative short rate r is small enough, a higher value of a leads to a higher bond price. This can be explained by the following: the parameter a manages the speed of mean reversion of the short

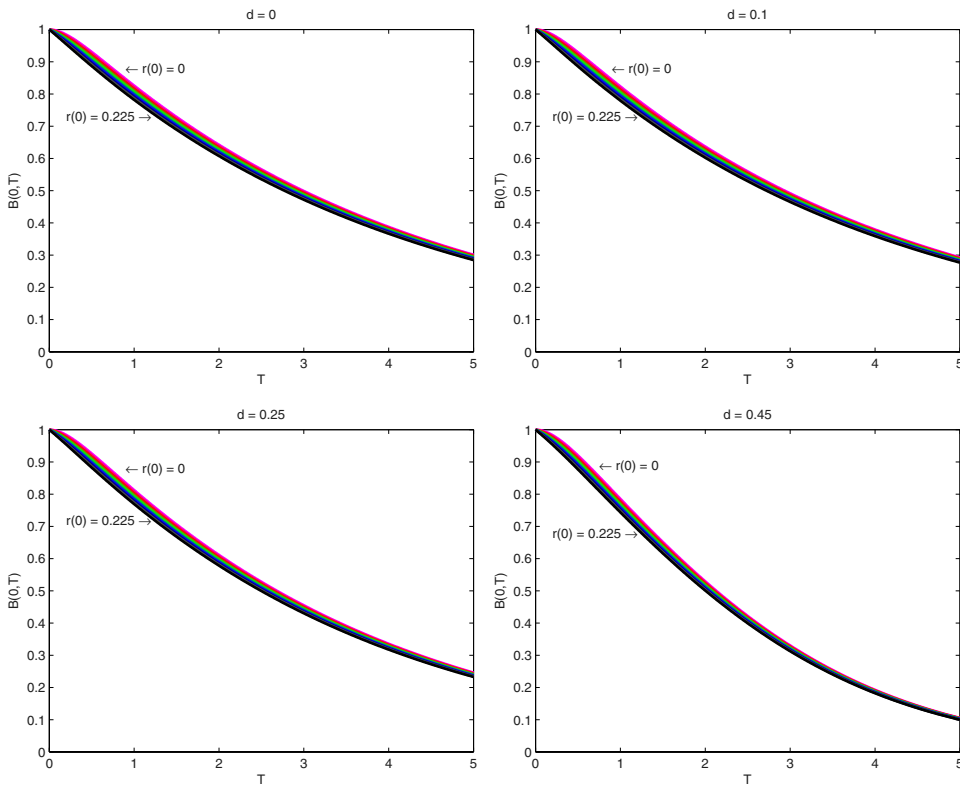


Fig. 2. Bond prices $B^{\text{Poi}}(0, T)$ in the fractional Poisson model for varying $r(0)$, maturity T and fractional parameter d , using constant coefficients $a = 4, k = 1, \sigma = 1$ and $\eta = 1$. In particular, $r(0)$ increases by steps of size 0.025 from 0 to 0.225.

rate. With high probability any short time divergence from the mean will lead to a higher short rate (since negative values occur only with small probability) and therefore to a lower bond price. High values of a lead to a stronger mean-reversion and therefore the impact of such a potential divergence will be small and *vice versa*.

The suddenly increasing prices are explained by the positive probability of getting a negative short rate. The variance of r increases with the fractional parameter and therefore the probability of negative values also rises. As a consequence, bond prices tend to get higher for longer maturities. Combined with a weaker mean reversion (i.e. small values of a) this effect is even stronger.

Since in the fractional Poisson model the short rate is always positive the influence of a is more straightforward as Fig. 4 shows: a higher value of a leads *ceteris paribus* to a higher bond price. The calculation of the derivative with

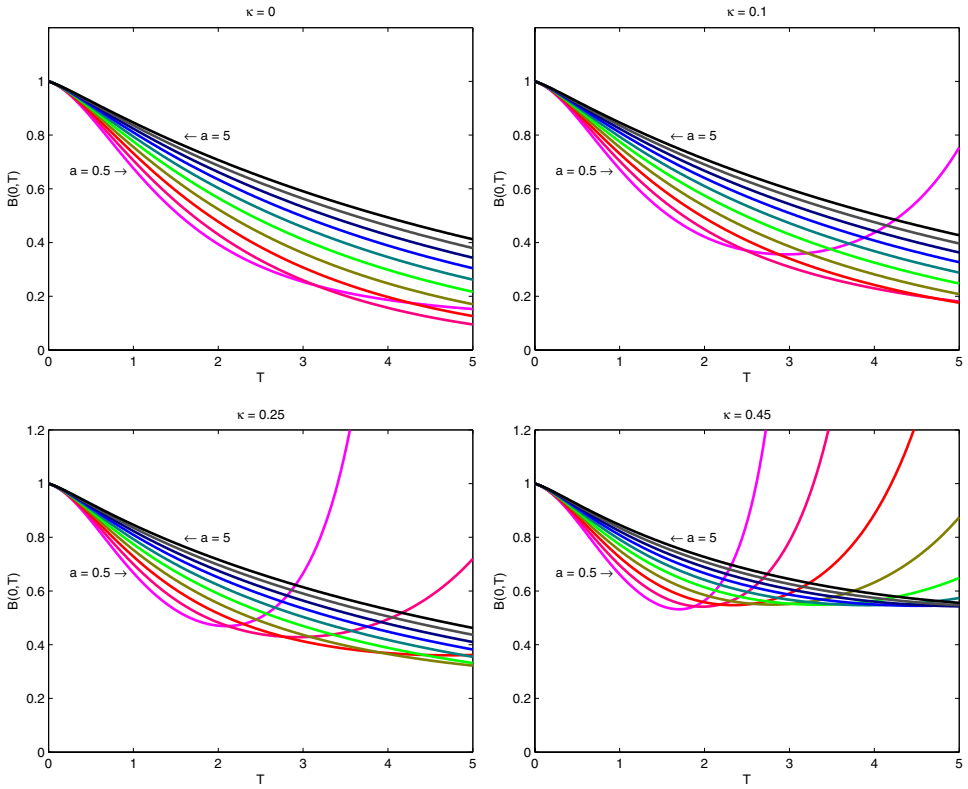


Fig. 3. Bond prices $B^{\text{fBm}}(0, T)$ in the fractional Brownian model for varying a , maturity T and fractional parameter d , using constant coefficients $r(0) = 0.1$, $k = 1$ and $\sigma = 1$. $d = 0$ corresponds to the classical Brownian Vasicek model. In particular a increases by steps of size 0.5 from 0.5 to 5.

respect to a is more complicated and follows:

$$\frac{\partial B_a^{\text{fBm}}(0, T)}{\partial a} = \left(-\frac{\partial D(0, T)}{\partial a} r(0) - k \frac{\partial \int_0^T D(v, T) dv}{\partial a} + \frac{\sigma^2}{2} \frac{\partial}{\partial a} \left\| z^d(D(\cdot, T) \mathbf{1}_{[0, T]}, v) \mathbf{1}_{[0, T]}(\cdot) \right\|^2 \right) \times B_a^{\text{fBm}}(0, T),$$

where we calculate using the classical rule for differentiation under the integral sign

$$\begin{aligned} & \frac{\partial}{\partial a} \left\| z^d(D(\cdot, T) \mathbf{1}_{[0, T]}, v) \mathbf{1}_{[0, T]}(\cdot) \right\|^2 \\ &= c_d^2 \frac{\partial}{\partial a} \int_0^T s^{-2d} \left(\int_s^{T^*} \frac{r^d D(r, T) \mathbf{1}_{[0, T]}(r)}{(r-s)^{1-d}} dr \right)^2 ds \end{aligned}$$

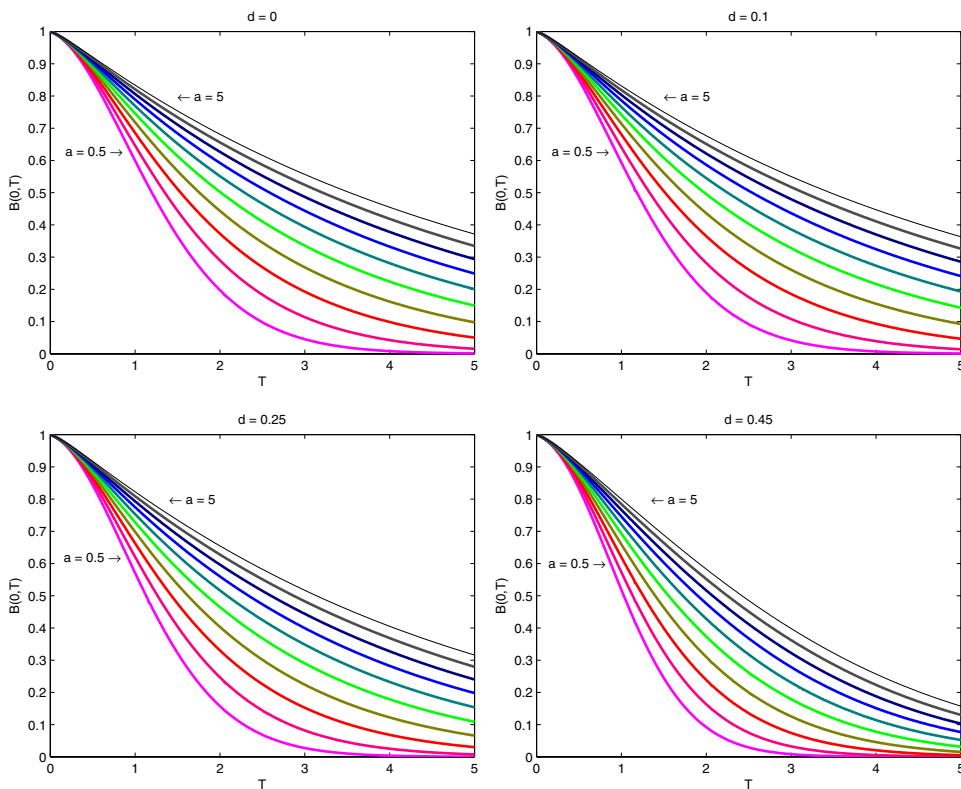


Fig. 4. Bond prices $B^{\text{Poi}}(0, T)$ in the fractional Poisson model for varying a , maturity T and fractional parameter d , using constant coefficients $r(0) = 0.1$, $k = 1$, $\sigma = 1$ and $\eta = 1$. In particular a increases by steps of size 0.5 from 0.5 to 5.

$$\begin{aligned}
 &= c_d^2 \int_0^T s^{-2d} \frac{\partial}{\partial a} \left(\int_s^{T^*} \frac{r^d D(r, T) \mathbf{1}_{[0, T]}(r)}{(r-s)^{1-d}} dr \right)^2 ds \\
 &= 2c_d^2 \int_0^T s^{-2d} \left(\int_s^{T^*} \frac{r^d D(r, T) \mathbf{1}_{[0, T]}(r)}{(r-s)^{1-d}} dr \right) \\
 &\quad \times \frac{\partial}{\partial a} \left(\int_s^{T^*} \frac{r^d D(r, T) \mathbf{1}_{[0, T]}(r)}{(r-s)^{1-d}} dr \right) ds \\
 &= 2c_d^2 \int_0^T s^{-2d} \left(\int_s^{T^*} \frac{r^d D(r, T) \mathbf{1}_{[0, T]}(r)}{(r-s)^{1-d}} dr \right) \\
 &\quad \times \left(\int_s^{T^*} \frac{r^d \frac{\partial}{\partial a} D(r, T) \mathbf{1}_{[0, T]}(r)}{(r-s)^{1-d}} dr \right) ds.
 \end{aligned}$$

We finally see that

$$\begin{aligned} & \frac{\partial}{\partial a} \| z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v)\mathbf{1}_{[0, T]}(\cdot) \|^2 \\ &= 2 \left\langle z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v)\mathbf{1}_{[0, T]}(\cdot), z^d\left(\frac{\partial}{\partial a}D(\cdot, T)\mathbf{1}_{[0, T]}, v\right)\mathbf{1}_{[0, T]}(\cdot) \right\rangle. \end{aligned}$$

In the fractional Poisson case we have

$$\begin{aligned} \frac{\partial B_a^{\text{Poi}}(0, T)}{\partial a} &= \left(-\frac{\partial D(0, T)}{\partial a}r(0) - k\frac{\partial \int_0^T D(v, T)dv}{\partial a} \right. \\ &\quad \left. + \eta\frac{\partial}{\partial a} \int_0^T \exp(-\sigma z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v))dv \right) \times B_a^{\text{fBm}}(0, T), \end{aligned}$$

where we obtain again by interchanging differentiation and integration

$$\begin{aligned} & \frac{\partial}{\partial a} \int_0^T \exp(-\sigma z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v))dv \\ &= \int_0^T \frac{\partial}{\partial a} \exp(-\sigma z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v))dv \\ &= -\sigma \int_0^T \frac{\partial}{\partial a} z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v) \exp(-\sigma z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v))dv. \end{aligned}$$

Therefore we have in total

$$\begin{aligned} \frac{\partial B_a^{\text{fBm}}(0, T)}{\partial a} &= \left(-\frac{\partial D(0, T)}{\partial a}r(0) - k \int_0^T \frac{\partial}{\partial a} D(v, T)dv \right. \\ &\quad \left. + \sigma^2 \left\langle z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v)\mathbf{1}_{[0, T]}(\cdot), z^d \right. \right. \\ &\quad \left. \left. \left(\frac{\partial}{\partial a}D(\cdot, T)\mathbf{1}_{[0, T]}, v \right)\mathbf{1}_{[0, T]}(\cdot) \right\rangle \right) \times B_a^{\text{fBm}}(0, T) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial B_a^{\text{Poi}}(0, T)}{\partial a} &= \left(-\frac{\partial D(0, T)}{\partial a}r(0) - k \int_0^T \frac{\partial}{\partial a} D(v, T)dv \right. \\ &\quad \left. - \eta\sigma \int_0^T \frac{\partial}{\partial a} z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v) \right. \\ &\quad \left. \times \exp(-\sigma z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v))dv \right) \times B_a^{\text{Poi}}(0, T). \end{aligned}$$

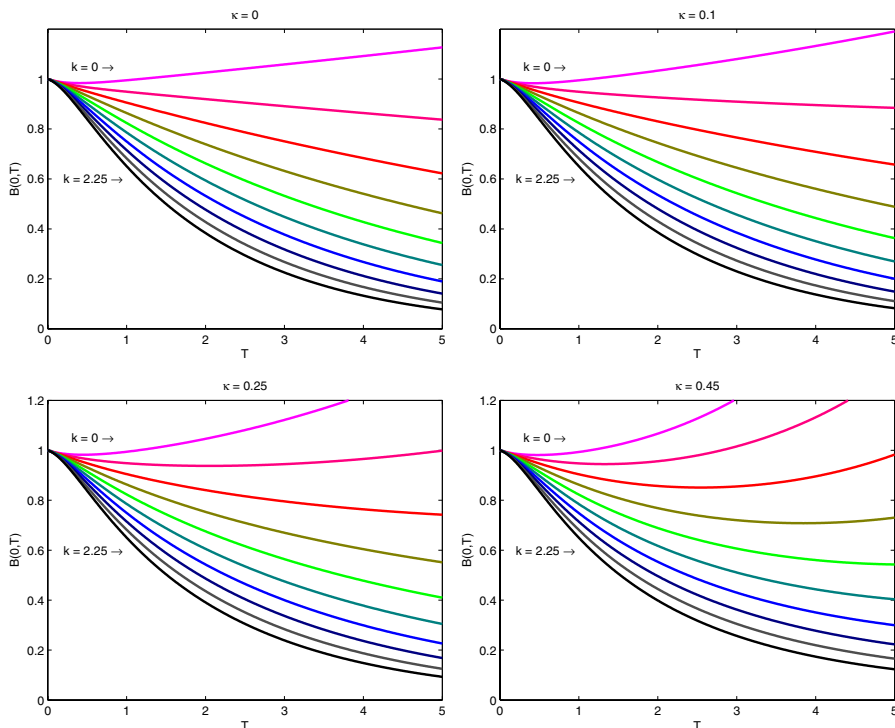


Fig. 5. Bond prices $B^{\text{fBm}}(0, T)$ in the fractional Brownian model for varying k , maturity T and fractional parameter d , using constant coefficients $r(0) = 0.1$, $a = 4$ and $\sigma = 1$. $d = 0$ corresponds to the classical Brownian Vasicek model. In particular k increases by steps of size 0.25 from 0 to 2.25.

4.3. Parameter sensitivity with respect to k

The impact of the parameter k is again a bit more straightforward. Figure 5 shows that a higher value of k results mostly in a lower bond price since for fixed a , the parameter k controls the long term mean of the process r . For longer maturities the influence of k becomes stronger. A special case are again the suddenly increasing bond prices.

As before they can be explained by the probability of negative values of the short rate. Lower values of k increase this probability. Of course this is not an issue in the fractional Poisson model, cf. Fig. 6.

The derivative with respect to k exists and is given by

$$\frac{\partial B_k^{\text{fBm}}(0, T)}{\partial k} = - \int_0^T D(v, T) dv \cdot B_k^{\text{fBm}}(0, T),$$

$$\frac{\partial B_k^{\text{Poi}}(0, T)}{\partial k} = - \int_0^T D(v, T) dv \cdot B_k^{\text{Poi}}(0, T).$$

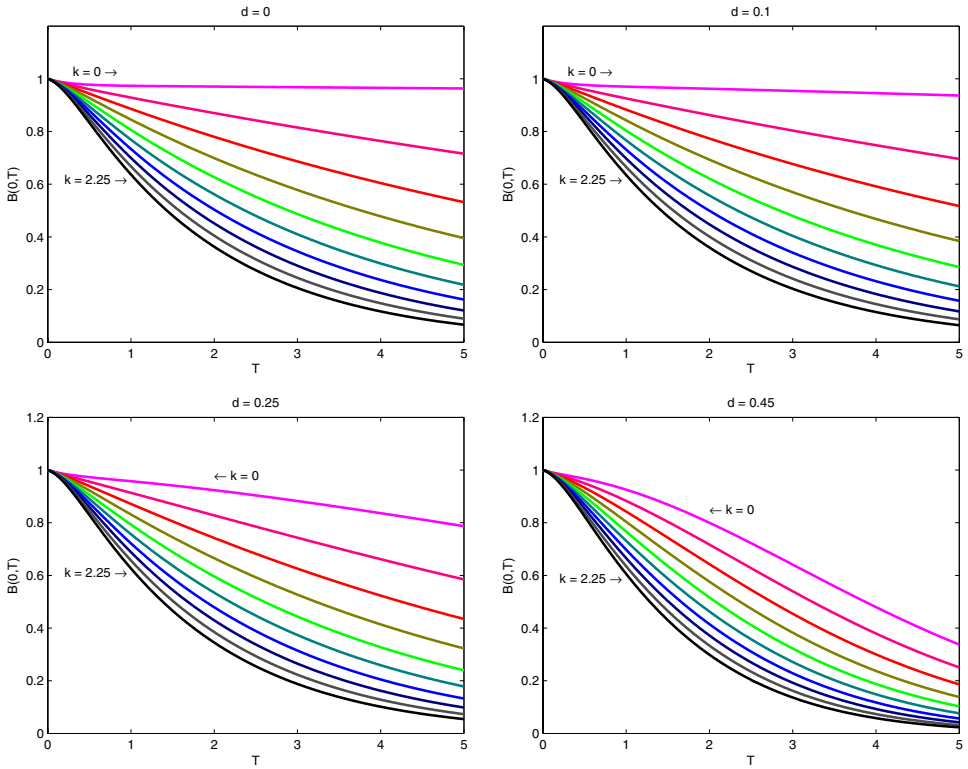


Fig. 6. Bond prices $B^{\text{Poi}}(0, T)$ in the fractional Poisson model for varying k , maturity T and fractional parameter d , using constant coefficients $r(0) = 0.1$, $a = 4$, $\sigma = 1$ and $\eta = 1$. In particular k increases by steps of size 0.25 from 0 to 2.25.

4.4. Parameter sensitivity with respect to σ

The parameter σ has a positive impact on the bond price in the fractional Brownian model, cf. Fig. 7. If σ equals zero the short rate r is deterministic and cannot become negative. Bond prices decrease with longer maturity. However if σ is positive and takes high values, the probability of r getting negative gets higher, which means that bond prices will be higher, too.

In the fractional Poisson model, cf. Fig. 8, the influence of the parameter σ is very different. Since the short rate is nonnegative the impact of σ is asymmetric. A higher value will increase the probability of seeing higher values of r which leads to a lower bond price. The derivative with respect to σ is given by

$$\frac{\partial B_{\sigma}^{\text{fBm}}(0, T)}{\partial \sigma} = \sigma \|z^d(D(\cdot, T)\mathbf{1}_{[0, T]}, v)\mathbf{1}_{[0, T]}(\cdot)\|^2 \cdot B_{\sigma}^{\text{fBm}}(0, T),$$

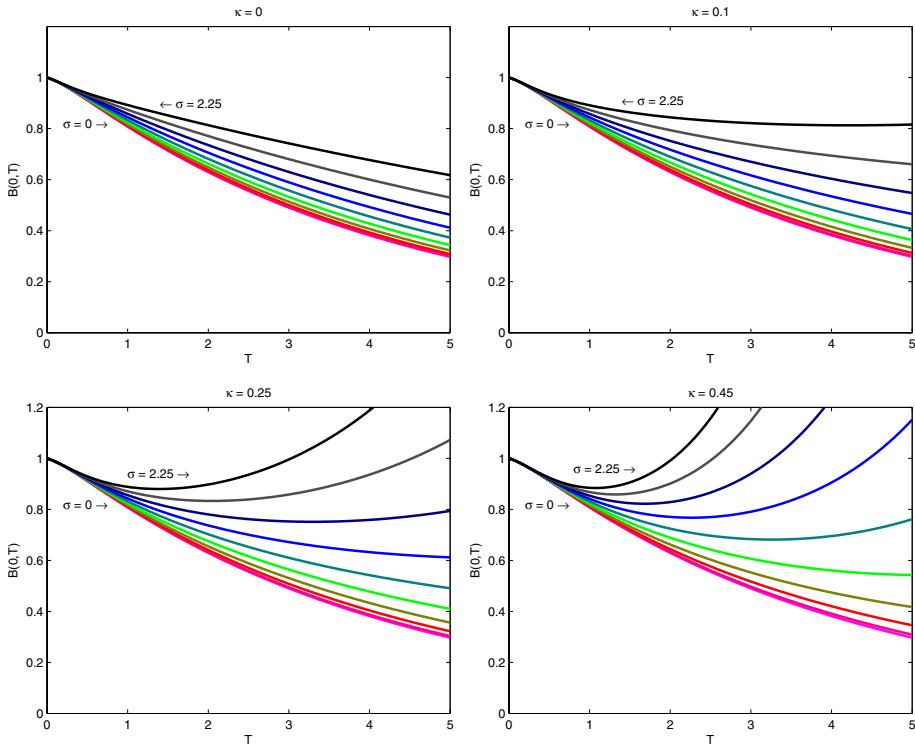


Fig. 7. Bond prices $B^{\text{fbm}}(0, T)$ in the fractional Brownian model for varying σ , maturity T and fractional parameter d , using constant coefficients $r(0) = 0.1$, $a = 4$ and $k = 1$. $d = 0$ corresponds to the classical Brownian Vasicek model. In particular σ increases by steps of size 0.25 from 0 to 2.25.

$$\frac{\partial B_{\sigma}^{\text{Poi}}(0, T)}{\partial \sigma} = -\eta \int_0^T z^d (D(\cdot, T) \mathbf{1}_{[0, T]}, v)) \times \exp(-\sigma z^d (D(\cdot, T) \mathbf{1}_{[0, T]}, v)) dv \cdot B_{\sigma}^{\text{Poi}}(0, T),$$

where we used the classical rule for differentiation under the integral sign in the fractional Poisson case.

4.5. Parameter sensitivity with respect to η

As can be seen in Fig. 9 the impact of the Poisson parameter η is always monotone: higher values of η lead to lower bond prices. The reason is that with increasing η the fractional Poisson subordinator driving the short rate process will have a higher upward drift. The derivative with respect to η is given by

$$\frac{\partial B_{\eta}^{\text{Poi}}(0, T)}{\partial \eta} = \int_0^T \exp(-\sigma z^d (D(\cdot, T) \mathbf{1}_{[0, T]}, v)) dv \cdot B_{\eta}^{\text{Poi}}(0, T).$$

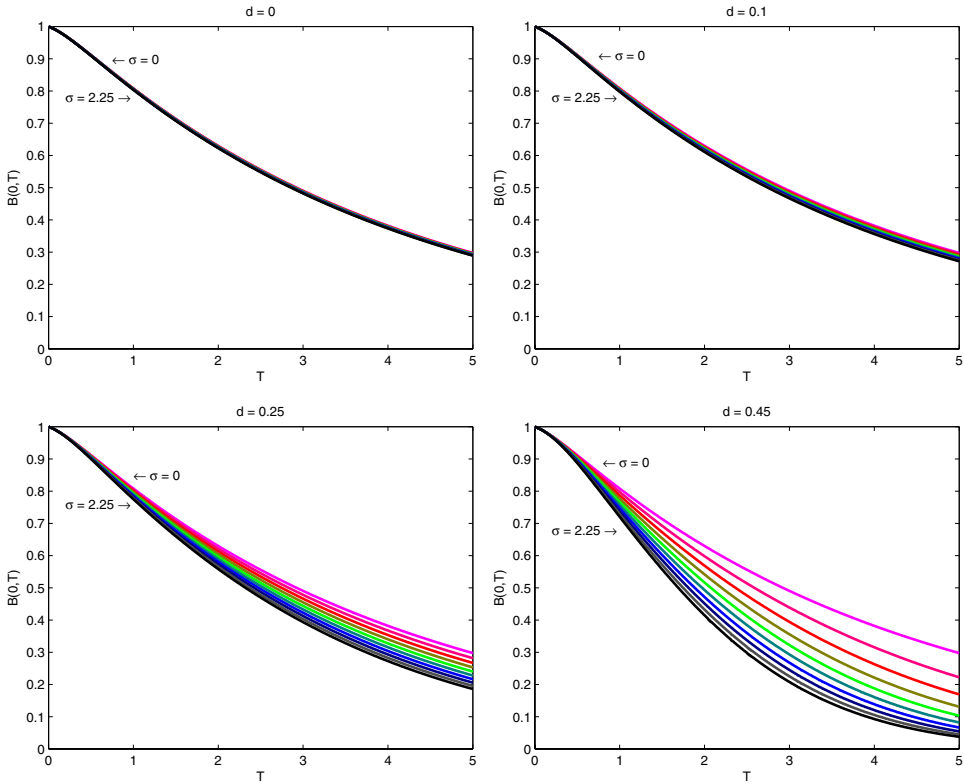


Fig. 8. Bond prices $B^{\text{Poi}}(0, T)$ in the fractional Poisson model for varying σ , maturity T and fractional parameter d , using constant coefficients $r(0) = 0.1$, $a = 4$, $k = 1$ and $\eta = 1$. In particular σ increases by steps of size 0.25 from 0 to 2.25.

5. Pricing Credit Default Swaps

A CDS is a bilateral contract that ensures compensation payment to a *protection buyer* in the case that the underlying entity defaults. In turn, the so-called *protection seller* receives a contractually fixed premium up to expiry or default. The payments made by the protection buyer are aggregated to the premium leg, whereas payments effected by the protection seller are part of the protection leg. For simplicity, we assume the notional of the contract to be one unit of money and focus on a running CDS contract with postponed payments. Thus, in the case of default at τ , $T_{i-1} < \tau \leq T_i$, $i \in \{1, \dots, m\}$, the compensation payment is unwound at T_i . The sequence $(T_i)_{i=1, \dots, m}$ refers to the schedule of the coupon or spread payments. The amount of these payments equals $\xi_i R$, where R symbolizes the spread rate and $\xi_i := T_i - T_{i-1}$. For example, if $(T_i)_{i=0, \dots, m}$ represents a grid with a uniform distance of three months, then ξ_i always equals $3/12$. In the case of

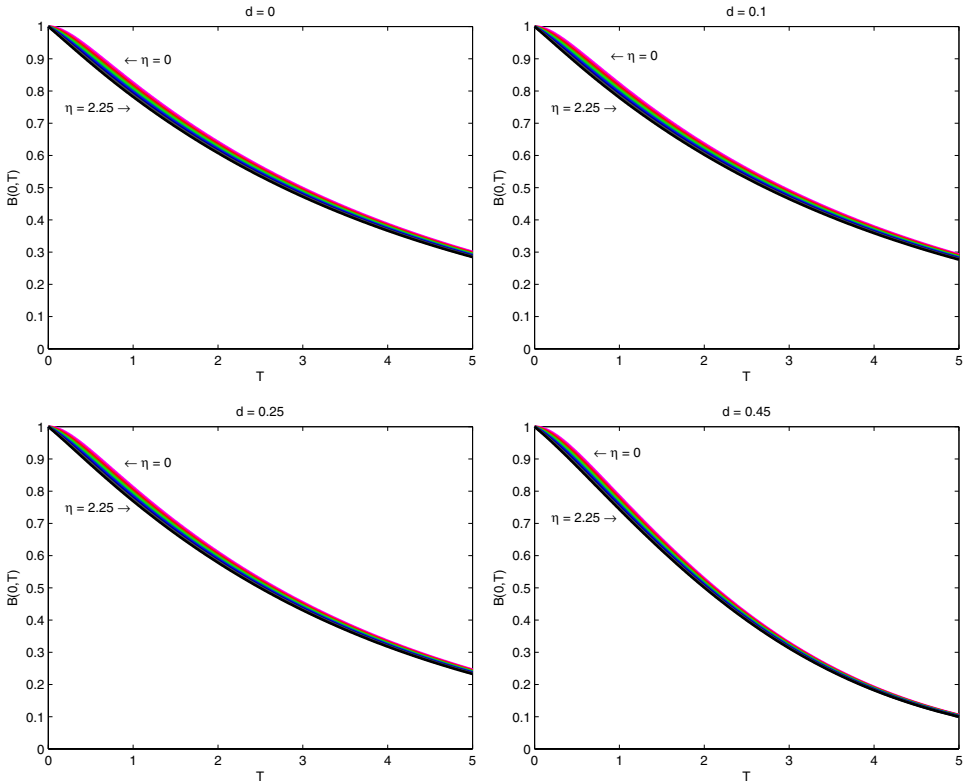


Fig. 9. Bond prices $B^{\text{Poi}}(0, T)$ in the fractional Poisson model for varying η , maturity T and fractional parameter d , using constant coefficients $r(0) = 0.1$, $a = 4$ and $k = 1$. In particular η increases by steps of size 0.25 from 0 to 2.25.

default, the protection seller has to pay the amount L to the protection buyer. Generally, this amount appears to be stochastic, but for reasons of simplicity it can also take a fixed value. Here, we proceed this way.

5.1. Pricing theorem

For better readability and understanding we will now describe the general approach to CDS pricing, cf. Filipovic (2009), Schönbucher (2003) or Brigo and Mercurio (2001) for details. From the perspective of a protection seller, the discounted payoff of a CDS contract at time t , $0 \leq t \leq T_m = T^*$, is given by

$$\Pi(t) := \sum_{i=1}^m e^{-\int_t^{T_i} r(s)ds} \xi_i R \cdot \mathbf{1}_{\{\tau > T_i\}} - \sum_{i=1}^m e^{-\int_t^{T_i} r(s)ds} L \cdot \mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} \quad (5.1)$$

as stated in Eq. (21.13) of Brigo and Mercurio (2001). The value of the CDS contract can now be calculated as the expectation under the pricing measure \mathcal{Q}

because it is a \mathcal{F}_{T^*} -measurable, integrable claim. Since its current value at time t is denoted by $\text{CDS}(t, R, L)$, we set

$$\text{CDS}(t, R, L) = E[\Pi(t) | \mathcal{F}_t].$$

Similar to Sec. 4.3, where we priced a (defaultable) zero coupon bond, we can apply Proposition 3.1 of Lando (1998) or the principle of Lemma 13.2 of Filipovic (2009), which implies that for a \mathcal{F}_{T^*} -measurable, integrable claim X we have the equality

$$E[X | \mathcal{F}_t] = \mathbf{1}_{\{\tau > t\}} \frac{E[X | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \tag{5.2}$$

with $0 \leq t \leq T^*$. In particular, for $0 \leq t \leq T \leq T^*$ and

$$X = e^{-\int_t^T r(s)ds} \mathbf{1}_{\{\tau > T\}}$$

i.e. the case of a defaultable zero coupon bond, we obtain

$$\begin{aligned} E[e^{-\int_t^T r(s)ds} \mathbf{1}_{\{\tau > T\}} | \mathcal{F}_t] &= \mathbf{1}_{\{\tau > t\}} \frac{E[e^{-\int_t^T r(s)ds} \mathbf{1}_{\{\tau > T\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \\ &= \mathbf{1}_{\{\tau > t\}} E[e^{-\int_t^T [r(s) + \lambda(s)]ds} | \mathcal{G}_t] = \bar{B}(t, T). \end{aligned}$$

Using Eq. (5.2), the value of the CDS contract with respect to the σ -algebra \mathcal{G}_t can be rewritten as follows:

$$\text{CDS}(t, R, L) = E[\Pi(t) | \mathcal{F}_t] = \mathbf{1}_{\{\tau > t\}} \frac{E[\Pi(t) | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)}, \quad 0 \leq t \leq T_m.$$

In the next step, we use the linearity of the conditional expectation to calculate

$$\begin{aligned} \text{CDS}(t, R, L) &= \mathbf{1}_{\{\tau > t\}} \frac{E[\Pi(t) | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \\ &= \mathbf{1}_{\{\tau > t\}} \frac{1}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \left\{ \sum_{i=1}^m \xi_i R \cdot E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{\tau > T_i\}} | \mathcal{G}_t] \right. \\ &\quad \left. - \sum_{i=1}^m L \cdot E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} | \mathcal{G}_t] \right\} \\ &= \mathbf{1}_{\{\tau > t\}} \left\{ R \cdot \sum_{i=1}^m \xi_i \frac{E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{\tau > T_i\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \right. \\ &\quad \left. - L \cdot \sum_{i=1}^m \frac{E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \right\} \end{aligned}$$

$$= R \cdot \sum_{i=1}^m \xi_i \bar{B}(t, T_i) - \mathbf{1}_{\{\tau > t\}} L \cdot \sum_{i=1}^m \frac{E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)}.$$
(5.3)

However, the conditional expectation

$$E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} | \mathcal{G}_t]$$

is difficult to determine, due to characteristics of the embedded default indicator function. Therefore, we apply

$$\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} = 1 - \mathbf{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{\tau \leq T_{i-1}\}}.$$

Hence, we get by setting $B(t, T_i) := E[e^{-\int_t^{T_i} r(s) ds} | \mathcal{G}_t]$

$$\begin{aligned} & \mathbf{1}_{\{\tau > t\}} \frac{E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \\ &= \mathbf{1}_{\{\tau > t\}} \frac{E[e^{-\int_t^{T_i} r(s) ds} | \mathcal{G}_t] - E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{\tau > T_i\}} | \mathcal{G}_t] - E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)} \\ &= \mathbf{1}_{\{\tau > t\}} e^{\int_0^t \lambda(s) ds} B(t, T_i) - \bar{B}(t, T_i) - \mathbf{1}_{\{\tau > t\}} \frac{E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)}. \end{aligned}$$

We wish to stress that these formulas also apply if $t > T_i$. Yet, in this case, the quantities $B(t, T_i)$ and $\bar{B}(t, T_i)$ can no longer be interpreted as bond prices since they do not reflect discounting but accruing interest in the following sense: Let $t > T_i$ and condition on the set $\{\tau > t\}$, then we have formally

$$B(t, T_i) = E[e^{-\int_t^{T_i} r(s) ds} | \mathcal{G}_t] = E[e^{\int_{T_i}^t r(s) ds} | \mathcal{G}_t] = e^{\int_{T_i}^t r(s) ds} E[1 | \mathcal{G}_t] = e^{\int_{T_i}^t r(s) ds}$$

and

$$\bar{B}(t, T_i) = E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{\tau > T_i\}} | \mathcal{F}_t] = E[e^{\int_{T_i}^t r(s) ds} | \mathcal{F}_t] = e^{\int_{T_i}^t r(s) ds}.$$

Finally, we obtain the following well-known pricing theorem:

Theorem 5.1. *In the situation above, we have for $0 \leq t \leq T_m$:*

$$\begin{aligned} \text{CDS}(t, R, L) &= \sum_{i=1}^m [(\xi_i R + L) \bar{B}(t, T_i) - \mathbf{1}_{\{\tau > t\}} L \cdot e^{\int_0^t \lambda(s) ds} B(t, T_i)] \\ &+ \mathbf{1}_{\{\tau > t\}} L \cdot \sum_{i=1}^m \frac{E[e^{-\int_t^{T_i} r(s) ds} \mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t]}{\mathcal{Q}(\tau > t | \mathcal{G}_t)}. \end{aligned}$$
(5.4)

The first part of this formula can be computed easily. However, the term

$$E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t]$$

is more problematic since we do not have an explicit representation in terms of bonds.

5.2. CDS rate process

As it is often market convention, the fixed rate R , which the protection buyer has to pay, is chosen such that the value of the respective CDS contract is zero. Therefore, we define the CDS rate process $(R(t))_{0 \leq t \leq T_m}$ by

$$\text{CDS}(t, R(t), L) = 0, \quad 0 \leq t \leq T_m.$$

As an immediate consequence, we get

$$R(t) = L \cdot \frac{\sum_{i=1}^m \{e^{\int_0^t \lambda(s)ds} \{B(t, T_i) - E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t]\} - \bar{B}(t, T_i)\}}{\sum_{i=1}^m \xi_i \bar{B}(t, T_i)}$$

on the set $\{\tau > t\}$. Since the definition on $\{\tau \leq t\}$ does not really matter, we can also formally set it to the above expression. As mentioned, we do not have an explicit representation for

$$E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t]$$

in our fractional credit market. A possible approach would be via simulation, where we would simulate paths of the underlying MG-fLp and estimate the conditional expectation given above. However, to obtain analytical expressions for the characteristics of CDS rate term structures in our fractional setting, we assume independence of short rate and default time from one another. This is equivalent to the assumption of independent short and hazard rate processes. Hence, we proceed similarly to Jarrow *et al.* (1997) and Jarrow and Turnbull (1995), who assumed that the stochastic process for the default-free interest rate and the intensity-based bankruptcy process are stochastically independent under the chosen risk-neutral measure. The latter process is represented by the stochastic variable τ that denotes the random time at which bankruptcy occurs. Jarrow *et al.* (1997), as well as Jarrow and Turnbull (1995), also justified their line of action by the fact that this approach entails a simplified analysis and the capability of deriving further (numerical) results. Moreover, they claimed that under certain conditions this assumption also implies independence among the corresponding real-world

(physical) quantities. We use the proposed independence assumption to calculate further:

$$\begin{aligned} E[e^{-\int_t^{T_i} r(s)ds} \mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t] &= E[e^{-\int_t^{T_i} r(s)ds} | \mathcal{G}_t] \cdot E[\mathbf{1}_{\{\tau \leq T_{i-1}\}} | \mathcal{G}_t] \\ &= B(t, T_i) E[1 - \mathbf{1}_{\{\tau > T_{i-1}\}} | \mathcal{G}_t] \\ &= B(t, T_i) \{1 - E[e^{-\int_t^{T_{i-1}} \lambda(s)ds} | \mathcal{G}_t] e^{-\int_0^t \lambda(s)ds}\}. \end{aligned}$$

Define for $0 \leq t \leq T \leq T^*$

$$H(t, T) := E[e^{-\int_t^T \lambda(s)ds} | \mathcal{G}_t].$$

As a result, we get

$$R(t) = L \cdot \frac{\sum_{i=1}^m \{e^{\int_0^t \lambda(s)ds} \{B(t, T_i) - B(t, T_i)[1 - H(t, T_{i-1})e^{-\int_0^t \lambda(s)ds}\}] - \bar{B}(t, T_i)\}}{\sum_{i=1}^m \xi_i \bar{B}(t, T_i)}.$$

Moreover, the price of a defaultable bond decomposes via

$$\begin{aligned} \bar{B}(t, T) &= \mathbf{1}_{\{\tau > t\}} E[e^{-\int_t^T [r(s) + \lambda(s)]ds} | \mathcal{G}_t] \\ &= \mathbf{1}_{\{\tau > t\}} E[e^{-\int_t^T r(s)ds} | \mathcal{G}_t] \cdot E[e^{-\int_t^T \lambda(s)ds} | \mathcal{G}_t] \\ &= \mathbf{1}_{\{\tau > t\}} B(t, T) \cdot H(t, T). \end{aligned}$$

Connecting these results, we obtain for today's CDS rate

$$\begin{aligned} R(0) &= L \cdot \frac{\sum_{i=1}^m \{B(0, T_i) - B(0, T_i)[1 - H(0, T_{i-1})] - \bar{B}(0, T_i)\}}{\sum_{i=1}^m \xi_i \bar{B}(0, T_i)} \\ &= L \cdot \frac{\sum_{i=1}^m B(0, T_i)[H(0, T_{i-1}) - H(0, T_i)]}{\sum_{i=1}^m \xi_i B(0, T_i) H(0, T_i)}. \end{aligned} \tag{5.5}$$

5.3. Term structure analysis

To determine CDS rate term structures, we apply Theorem 3.4 and, by way of example, focus on the *fractional Brownian* and the *fractional Poisson credit market*. We assume constant coefficient functions and set furthermore $\theta = (1, 0)^\top$ and $\phi = (0, 1)^\top$. Moreover, we impose the following restrictions:

$$a_{11}, a_{22}, k_1, k_2, \sigma_{11}, \sigma_{22} > 0 \quad \text{and} \quad a_{12} = a_{21} = \sigma_{12} = \sigma_{21} = 0.$$

Example 5.2. [Fractional Brownian market with credit risk] For $d = 2$, let L^d be a fractional Brownian motion with independent marginal processes. Thus, we can calculate

$$\begin{aligned} H(0, T) &= E[e^{-\int_0^T \lambda(s) ds}] \\ &= \exp \left\{ -D_{22}(0, T)\lambda(0) - k_2 \int_0^T D_{22}(v, T) dv \right. \\ &\quad \left. + \frac{\sigma_{22}^2}{2} \| z^{d(2)}(D_{22}(\cdot, T)\mathbf{1}_{[0, T]}, v)\mathbf{1}_{[0, T]}(\cdot) \|^2 \right\}. \end{aligned}$$

In the fractional Brownian setup, stochastic changes happen symmetrically. Yet, the exponential function of the bond pricing formula (3.3) gives more weight to an increment of the negative input argument than to a decrement. This asymmetry emerges all the more, the more variability a fractional Brownian motion exhibits. As can be immediately seen from Proposition 2.5, the variance of a fractional Lévy process enlarges as the fractional parameter scales up. Due to the independence assumption, the effects of varying $d(1)$ and $d(2)$ can be discussed separately. Increasing the fractional parameter $d(2)$ entails an amplified variance of the associated fractional Brownian motion and therefore also of the mean-reverting process describing λ . Hence, the cited asymmetry caused by the exponential function gets a greater impact. Keeping in mind that there is no risk sensitivity under the risk-neutral measure, the spread rate has to decrease as a consequence of the lower threat of default. This is depicted by Fig. 11. It is also shown that in the extreme case of $d(2) = 0.45$ the CDS rate curve declines after a certain period of time.

However, in line with Sarig and Warga (1989), humped spread curves are typical for CDS contracts with non-investment grade underlyings and also can be reproduced by our fractional Brownian model.

Concerning the effects of varying $d(1)$, we have to briefly review the properties of a CDS contract. Under the assumption of an investment grade underlying, expected premium payments only decrease very slowly and, in the limit, can be regarded as constant. By contrast, expected protection payments initially are vanishingly low but rise over the long run. Similar to the hazard rate, increasing the parameter $d(1)$ in the fractional Brownian setting entails an amplified weighting of states characterized by a low or even negative short rate. As a consequence, the high values of expected protection payments near to maturity get a greater impact, and hence, as shown in Fig. 10, spread rates increase.

Example 5.3. [Fractional Poisson credit market]. Assume an underlying bivariate Poisson MG-fLp with independent marginal processes and jump

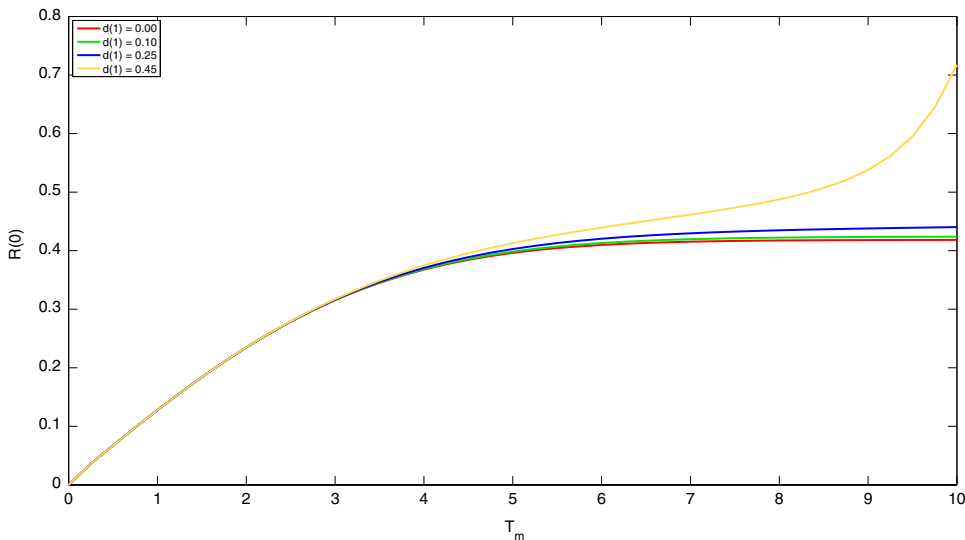


Fig. 10. CDS rate $R(0)$ with quarterly payments, i.e. $\xi_i = 0.25$, in the fractional Brownian market (Example 5.2) for varying $d(1)$ and maturity T_m , using constant coefficients $a_{11} = 0.01$, $a_{22} = 0.02$, $k_1 = 0.0025$, $k_2 = 0.05$, $\sigma_{11} = 0.1$, $\sigma_{22} = 0.075$, $(r(0), \lambda(0))^T = (0.01, 0.005)^T$ and $d(2) = 0.25$. The sharp increasing graph ($d(1) = 0.45$) is explained due to the fact that short and default rate are negative with positive probability.

intensities η_1 and η_2 . At first glance, it might be confusing that CDS rate curves in the Brownian and the Poisson setting reveal contrary patterns. This, however, can be ascribed to the very different characteristics of the processes controlling the interest as well as the hazard rate (cf. Sec. 4). In the fractional Poisson credit market, the driving processes are almost surely increasing, and therefore, in general, only positive movements are incited by the stochastic part of the mean reversion process. In contrast to the Brownian case, the mean reversion level marks a lower boundary for the modeled quantity. Increasing the fractional parameter means increasing the variance and thus, on average, a higher level of short as well as hazard rates can be observed. As stated above, higher hazard rates imply a heightened probability of default, which in turn causes a rising spread rate. Higher short rates put less weight on protection payments usually getting significant at the end of maturity. Consequently, the spread rate of the default swap contract declines. Figures 12 and 13 visualize the corresponding CDS rate term structures.

5.4. Term-structure calibration

At the end of this section, we want to fit our model-implied CDS term structure to real market data. Therefore we obtained end-of-day CDS rates for various

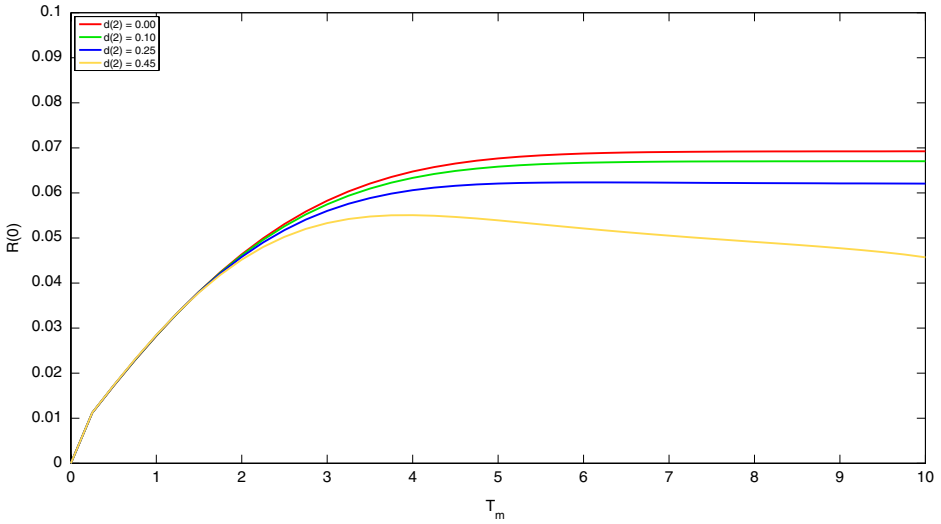


Fig. 11. CDS rate $R(0)$ with quarterly payments, i.e. $\xi_i = 0.25$, in the fractional Brownian market (Example 5.2) for varying $d(2)$ and maturity T_m , using constant coefficients $a_{11} = 0.01$, $a_{22} = 0.02$, $k_1 = 0.0025$, $k_2 = 0.05$, $\sigma_{11} = 0.1$, $\sigma_{22} = 0.075$, $(r(0), \lambda(0))^T = (0.01, 0.005)^T$ and $d(1) = 0.25$. The graph for $d(2) = 0.45$ is explained due to the fact that short and default rate are negative with positive possibility. However a CDS curve that suddenly declines again after a certain time is often observed in real market data. Therefore the potential negative hazard rate might actually be very useful in obtaining realistic market models.

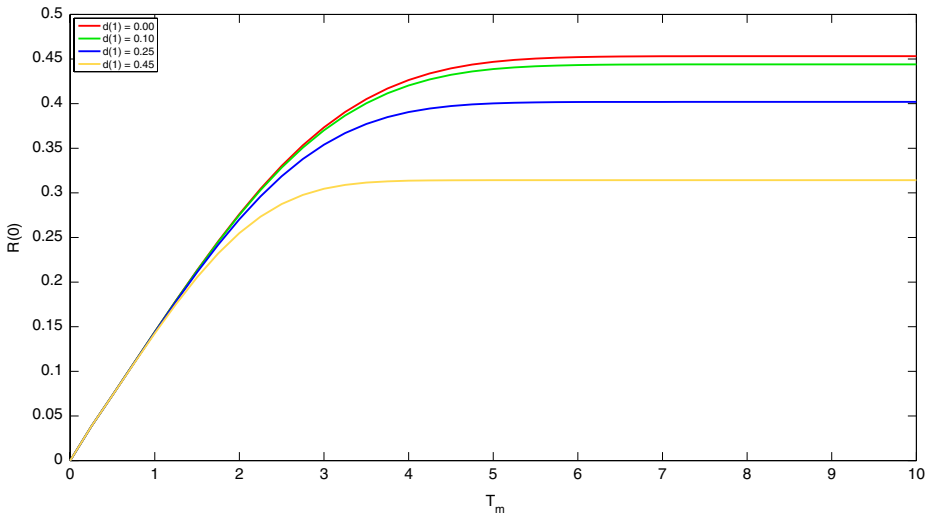


Fig. 12. CDS rate $R(0)$ with quarterly payments, i.e. $\xi_i = 0.25$, in the fractional Poisson market (Example 5.3) for varying $d(1)$ and maturity T_m , using constant coefficients $a_{11} = 0.01$, $a_{22} = 0.02$, $k_1 = 0.0025$, $k_2 = 0.05$, $\sigma_{11} = 0.1$, $\sigma_{22} = 0.075$, $(\eta_1, \eta_2)^T = (3, 3)^T$, $(r(0), \lambda(0))^T = (0.01, 0.005)^T$ and $d(2) = 0.25$.

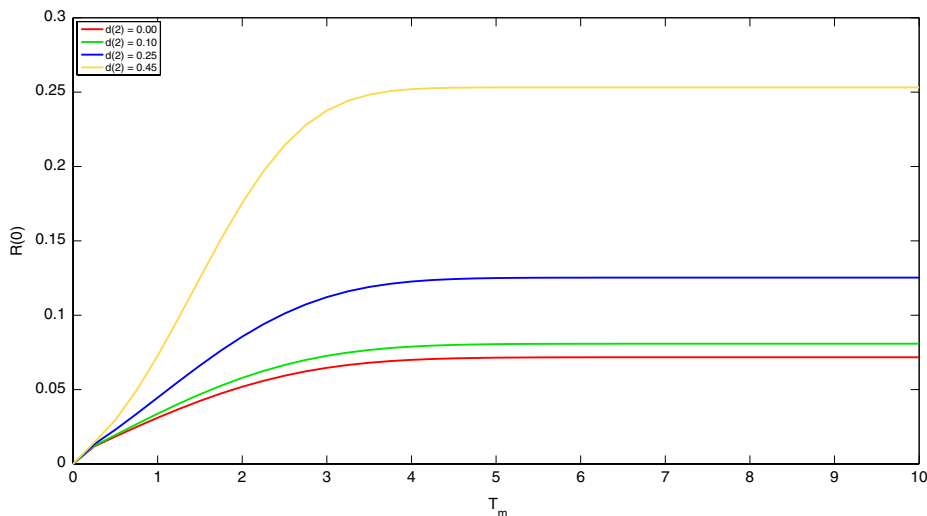


Fig. 13. CDS rate $R(0)$ with quarterly payments, i.e. $\xi_i = 0.25$, in the fractional Poisson market (Example 5.3) for varying $d(2)$ and maturity T_m , using constant coefficients $a_{11} = 0.01$, $a_{22} = 0.02$, $k_1 = 0.0025$, $k_2 = 0.05$, $\sigma_{11} = 0.1$, $\sigma_{22} = 0.075$, $(\eta_1, \eta_2)^\top = (3, 3)^\top$, $(r(0), \lambda(0))^\top = (0.01, 0.005)^\top$ and $d(1) = 0.25$.

constituents of the German stock index DAX and several maturities on 1 October 2014 using Thomson Reuters Datastream. These are shown in Fig. 14. Given the fact that short-term riskless rates in the Eurozone are currently slightly negative, we choose the fractional Brownian setting. Parameters are obtained by a least

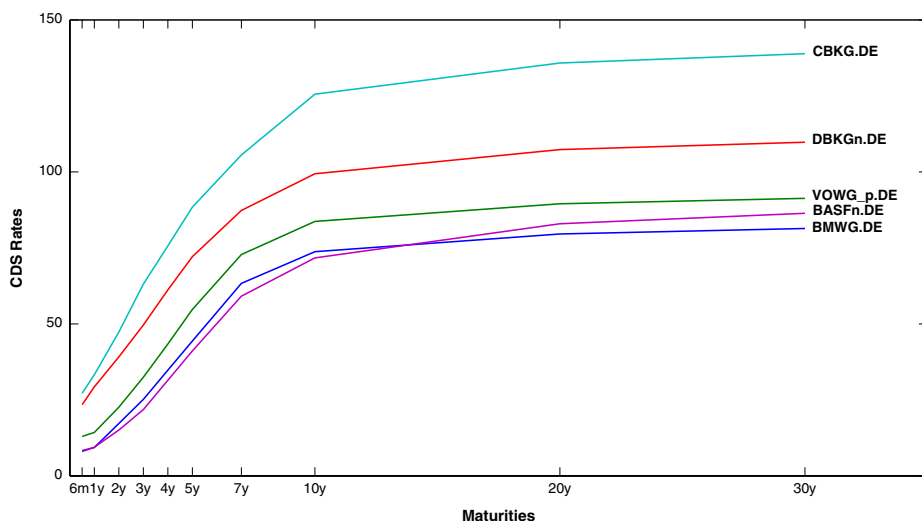


Fig. 14. CDS rates in basis points for various maturities and underlyings from the German DAX index on 1 October 2014. Source: Thomson Reuters Datastream.

Table 1. Least square calibration of model parameters in the Brownian setups using Eq. (5.5). Non-defaultable zero bond prices were obtained via the respective ECB AAA average bond yields from Thompson Reuters Datastream.

Underlying and Parameters		Brownian Market	Fractional Brownian Market
BMWG.DE	$\lambda(0)$	-0.0005	-0.0002
	a_{22}	0.2942	0.0977
	k_2	0.0030	0.0024
	σ_{22}	$2.2081 \cdot 10^{-08}$	0.0080
	$d(2)$	—	0.2392
	Sum of Squared Errors	$2.3027 \cdot 10^{-06}$	$1.2364 \cdot 10^{-06}$
VOWG_p.DE	$\lambda(0)$	-0.0002	0.0001
	a_{22}	0.3154	0.1054
	k_2	0.0035	0.0028
	σ_{22}	$1.5212 \cdot 10^{-08}$	0.0095
	$d(2)$	—	0.2187
	Sum of Squared Errors	$2.4721 \cdot 10^{-06}$	$1.2728 \cdot 10^{-06}$
DBKGn.DE	$\lambda(0)$	0.0011	0.0014
	a_{22}	0.3338	0.1160
	k_2	0.0043	0.0033
	σ_{22}	$1.5232 \cdot 10^{-08}$	0.0128
	$d(2)$	—	0.1524
	Sum of Squared Errors	$1.2107 \cdot 10^{-06}$	$4.9681 \cdot 10^{-07}$
CBKG.DE	$\lambda(0)$	0.0011	0.0015
	a_{22}	0.3217	0.1132
	k_2	0.0053	0.0041
	σ_{22}	$1.4145 \cdot 10^{-08}$	0.0141
	$d(2)$	—	0.1545
	Sum of Squared Errors	$1.4811 \cdot 10^{-06}$	$4.9718 \cdot 10^{-07}$
BASFn.DE	$\lambda(0)$	-0.0004	-0.0002
	a_{22}	0.2259	0.0728
	k_2	0.0025	0.0021
	σ_{22}	$1.8582 \cdot 10^{-08}$	0.0064
	$d(2)$	—	0.2490
	Sum of Squared Errors	$1.9437 \cdot 10^{-06}$	$1.1641 \cdot 10^{-06}$
ECB AAA-bonds	$r(0)$	-0.0118	-0.0060
	a_{11}	0.2301	0.0107
	k_1	0.0057	0.0036
	σ_{11}	$8.0033 \cdot 10^{-9}$	0.0066
	$d(1)$	—	0.2489
	Sum of Squared Errors	$7.7897 \cdot 10^{-4}$	$4.3127 \cdot 10^{-5}$

square calibration — additionally, this procedure is carried out for the special case $d(1) = d(2) = 0$ representing the classical non-fractional setup. For simplicity, a day count convention of 30/360 and quarterly payments were assumed for the CDS contract specifications. The results can be found in Table 1. Considering the sum of squared errors, the fractional setting seems to be better fitting to the data as the classical one. We want to stress that this should be considered a first look as a more detailed empirical study is beyond the scope of this paper. However, our findings seem to further support theories of long memory in hazard rates.

6. Conclusion

We used fractional Lévy processes defined via convolution of classical Lévy processes with independent increments and Molchan–Golosov kernels, leading to models that are able to capture long range dependence. Applying known results about fractional credit markets, we carried out a detailed sensitivity analysis of fractional bond prices in a fractional Brownian and a fractional Poisson setting. We calculated the derivatives with respect to the individual parameters which can be interpreted as a kind of sensitivity measure similar to the greeks in the classical Black-Scholes market. Afterwards we considered the pricing of CDS contracts and derived an explicit formula for the CDS rate. In the end, we were able to calculate rates for various maturities in a fractional Brownian and a fractional Poisson market and to analyze the impact of potential long range dependence. Furthermore, we carried out a brief empirical study to compare the fit in the fractional setting to the one in the classical models. Our findings here seem to strengthen conjectures of long memory in default rates, although a more thorough analysis should be the aim of future research.

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